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Positivity of the cotangent bundle

Talk I

$/\mathbb{C}$

I. Motivation

MMP is based on the positivity of canonical divisor

$$K_X = \det \Omega_X$$

Cotangent bundle Ω_X is also a "canonical object".
Why don't we use it to classify varieties?

- technical difficulties
- not much to show (many manifolds have K_X nef, but Ω_X nef is much more restrictive)

However, in the right setup, positivity of Ω_X becomes a powerful tool.

Ex: X projective manifold with $K_X \simeq \mathcal{O}_X$

Beauville-Bogomolov $\exists\ X' \to X$ étale finite s.t.

$$\boxed{X' = \underbrace{A}_{\text{abelian}} \times \prod \underbrace{Y_i}_{\text{Calabi-Yau}} \times \prod \underbrace{Z_j}_{\text{Hyperkähler}}}$$

Classical proof based on Yau's theorem
+ differential geometry

Modern proof (i.e. works also for klt spaces):

Case distinction:

1st case: T_X stable for all polarisations

$\overset{\text{Analysis}}{\Longrightarrow}$ X is C-Y, Hyperkähler or an elliptic curve

2nd case: T_X is not stable $\overset{\text{extra}}{\underset{\text{argument}}{\Longrightarrow}}$ $\underbrace{\boxed{T_X = V_1 \oplus V_2}}_{\substack{\text{infinitesimal decomposition} \\ \text{Greb-Kebekus-Peternell}}}$

Hope: V_1 and V_2 are tangent sheaves of manifolds appearing in the decomposition

Tool to realise the hope: show algebraic integrability of V_1 and V_2.

Plan of lecture series:

- Stability of vector bundles and sheaves
- Positivity of vector bundles
- The case $c_1 \leq 0$

References: Hartshorne
Lazarsfeld
Miyaoka-Peternell
Brunella

I. Stability of vector bundles on curves

C smooth projective curve

Given a line bundle $L \to C$ the degree $\deg L = \int_C c_1(L)$ says a lot about L: $\deg L > 0 \iff L$ is ample

$\deg L \geq 0 \iff L$ is nef

For a vector bundle $V \to C$ we set $\deg V = \deg(\det V)$, but this is a poor invariant. Example:

If L is ample and $V := L \oplus L^*$ then $\deg V = 0$

but V is far from being numerically trivial.

__Defn:__ Let $V \to C$ be a vector bundle. The slope of V is defined as $\mu(V) := \dfrac{\deg V}{\operatorname{rk} V}$

stable

We say that V is semistable if for every subbundle

$\updownarrow$

subsheaf

$K \overset{\subsetneq}{\subseteq} V$ one has $\mu(K) \overset{<}{\leq} \mu(V)$

__Ex 1:__ $L \to C$ ample sh $V := L \oplus \mathcal{O}_C \oplus L^*$

Then $\det V = \mathcal{O}_C$ so $\mu(V) = 0$

$\mathcal{O}_C \hookrightarrow V$ subbundle and $\mu(\mathcal{O}_C) = 0$ so V not stable

$L \hookrightarrow V$ " and $\mu(L) = \deg L > 0$

so V is not semistable

__Ex 2:__ A direct sum $V = V_1 \oplus V_2$ is never stable

It is semistable iff V_1, V_2 are semistable

and $\mu(V_1) = \mu(V_2)$

Very useful facts:

1) If V a vector bundle and L a line bundle, then

$$\mu(V\otimes L) = \frac{\deg(V\otimes L)}{\underset{\substack{\shortparallel \\ \operatorname{rk} V}}{\operatorname{rk}(V\otimes L)}} = \frac{\deg V + \operatorname{rk} V \cdot \deg L}{\operatorname{rk} V}$$

$$= \mu(V) + \deg L$$

2) V $\overset{\text{stable}}{\text{sstable}}$ $\Leftrightarrow$ $V\otimes L$ $\overset{\text{stable}}{\text{sstable}}$

Key lemma in stability theory:

$V \to C$ vector bundle. Then the set of slopes

$$\{\mu(K) \mid K \subset V \text{ subbundle}\}$$

is bounded from above. Moreover

$\exists!$ maximal subbundle $\mathcal{U} \subset V$ of largest slope

We call $\mathcal{U}$ the maximal destabiliser of V.

Proof of key lemma:

1) Boundedness

By theorem of Serre $\exists L \to C$ very ample line bundle

s.t. $V^* \otimes L$ is globally generated,

i.e. $\exists$ surjection $\mathcal{O}_C^{\oplus N} \to V^* \otimes L \to 0$

$\overset{\text{dualise}}{\Leftrightarrow}$ $V \otimes L^* \hookrightarrow \mathcal{O}_C^{\oplus N}$

If $K \subset V$ subbundle, then

$$K \otimes L^* \hookrightarrow V \otimes L^* \hookrightarrow \underbrace{\mathcal{O}_C^{\oplus N}}_{\substack{\text{sstable by exercise} \\ \text{slope} = 0}}$$

$$\Rightarrow \mu(K \otimes L^*) \leq 0$$

$$\overset{\text{fact 1}}{\Rightarrow} \mu(K) \leq \deg L$$

2) Maximal destabiliser

If $K_1, K_2 \subset V$ subbundles of maximal slope μ, then look at $\bar{K} \subset V$ the smallest subbundle that contains $K_1 + K_2$.

Using maximality of slope and extension

$$0 \to K_1 \cap K_2 \to K_1 \oplus K_2 \to K_1 + K_2 \to 0$$

we obtain $\mu(\bar{K}) = \mu$. Now restart with $\bar{K}$, this terminates since $\operatorname{rk} V$ is finite. ■

Consequence: any vector bundle $V \to C$ has a canonical filtration (= Harder-Narasimhan filtration)

$$0 = HN_0(V) \subset HN_1(V) \subset \dots \subset HN_{\ell-1}(V) \subset HN_\ell(V) = V$$

s.t. $Gr_i(V) := HN_i(V) / HN_{i-1}(V)$ is semistable

and

$$\mu(Gr_1(V)) > \dots > \mu(Gr_{\ell-1}(V)) > \mu(Gr_\ell(V)))$$

Idea of proof: $HN_1(V)$ = maximal destabiliser

Then do induction on the rank.

Ex: $V = L \oplus \mathcal{O}_C \oplus L^*$ L ample

i) if $K \hookrightarrow V$ subbundle of rank 1, then

$$\deg K \leq \deg L$$

Proof: otherwise

$$Hom(K, V) \simeq H^0(C, V \otimes K^*)$$

$$\simeq H^0(C, L \otimes K^* \oplus K^* \oplus L^* \otimes K^*)$$

$$= 0.$$

ii) if $K \hookrightarrow V$ subbundle of rank two

$\det K \hookrightarrow \Lambda^2 V$ subbundle

$$\Lambda^2(L\oplus\mathcal{O}_C\oplus L^*) = L\oplus\mathcal{O}_C\oplus L^*$$

so as before we show that $\deg K \leq \deg L$

$$\Rightarrow \mu(K) \leq \tfrac{1}{2}\deg L.$$

Conclusion: maximal slope is $\deg L$ and maximal destabiliser is $\underset{\underset{HN_1(V)}{\|}}{L} \hookrightarrow V$

iii) Maximal destabiliser of $\mathcal{O}_C\oplus L^*$ is $\mathcal{O}_C$

$$\Rightarrow HN_2(V) = L\oplus\mathcal{O}_C$$

$$HN_3(V) = L\oplus\mathcal{O}_C\oplus L^*$$

$$\begin{array}{ll} Gr_1 = L & \\ Gr_2 = \mathcal{O}_C & \Big\downarrow \text{ slope decreases} \\ Gr_3 = L^* & \end{array}$$

III. Stability of sheaves in higher dimension

More complicated: subbundles are not enough to understand stability of vector bundles.

Reasons: 1) If $V \to C$ vector bundle on a curve, then $\exists$ extension

$$0 \to K \to V \to Q \to 0$$

of vector bundles of smaller rank.

2) If $\mathcal{F} \to C$ is coherent sheaf that is torsion free, then it is locally free.

Not true if $\dim X \geq 2$: just take $\mathcal{F} = I_{\text{point}}$

Replacement: reflexive sheaves

Defn: X a normal variety, a coherent sheaf $\mathcal{F} \to X$ is reflexive, if

- $\mathcal{F}$ is torsion-free
- $\mathcal{F}$ is normal, i.e. $X' \subset X$ Zariski open and $X_0 \subset X'$ Zariski open s.t. $X' \setminus X_0 \subset X'$ has codim at least two then $H^0(X_0, \mathcal{F}) \simeq H^0(X', \mathcal{F})$.

Ex 1: locally free sheaves are reflexive

Ex 2: If $\dim X \geq 2$, then $\mathcal{I}_{\text{point}}$ is not reflexive.

Ex 3: If X is smooth and $\mathcal{F}$ is reflexive sheaf of rank one, then $\mathcal{F}$ is locally free.

Ex 4: $X \subset \mathbb{P}^3$ quadric cone $= \{ x_0^2 + x_1^2 + x_2^2 = 0 \}$ and $\ell \subset X$ a line through the vertex.

$\Rightarrow$ $\mathcal{O}_X(\ell)$ is reflexive, but not locally free.

For line bundles / Cartier divisors we know how to define intersection numbers.

For reflexive sheaves of rank one this is not always possible.

Defn: X normal projective variety of dimension n.

$\mathcal{F} \to X$ reflexive sheaf of rank one.

If $H \to X$ is a line bundle, then

$$\underbrace{\mathcal{F}} . \underbrace{H}^{n-1} = \sum a_i \int_{D_i} c_1(H)^{n-1}$$

where $\sum a_i D_i$ is a Weil divisor s.t.

$$\mathcal{F} \simeq \mathcal{O}_X\left(\sum a_i D_i\right)$$

Defn: X normal, $\mathcal{F} \to X$ torsion-free coherent sheaf

We set $\det \mathcal{F} = \underbrace{(\Lambda^{\mathrm{rk}\,\mathcal{E}} \mathcal{F})^{**}}_{\text{reflexive sheaf of rank one.}}$

Defn: Fix X normal projective of dimension n.

$H \to X$ ample line bundle

If $\mathcal{F} \to X$ is torsion-free sheaf, then the H-slope is defined as $\mu_H(\mathcal{F}) = \dfrac{\det \mathcal{F} \cdot H^{n-1}}{\mathrm{rk}\, \mathcal{F}}$

Defn: X normal variety, $\mathcal{F} \to X$ torsion-free sheaf

A subsheaf $K \subset \mathcal{F}$ is saturated iff the quotient $\mathcal{F}/K$ is torsion-free.

Remark: If C smooth curve, $\mathcal{F}$ a vector bundle.

A subsheaf $K \subset \mathcal{F}$ is saturated iff it is a subbundle.

Def. X normal proj, $H \to X$ ample. H-stable
$\mathcal{F} \to X$ torsion-free is H-sstable if
for every saturated subsheaf $K \overset{\subsetneq}{\subset} \mathcal{F}$ one has

$$\mu_H(K) \overset{<}{\leq} \mu_H(\mathcal{F})$$

Ex: $T_{\mathbb{P}^2}$ is H-stable ($H = \mathcal{O}_{\mathbb{P}^2}(1)$)

Proof: If $K \subset T_{\mathbb{P}^2}$ saturated subsheaf of rank 1,
then it is reflexive

⌜ $\mathcal{F}$ reflexive, $K \subset \mathcal{F}$ s.t. $\mathcal{F}/K$ is torsion-free,
then K is reflexive ⌟

$\overset{\mathbb{P}^2 \text{ smooth}}{\Longrightarrow}$ K is locally free of rank 1, so $K \simeq \mathcal{O}_{\mathbb{P}^2}(d)$

Euler sequence:

$$\begin{tikzcd}
 & & & \mathcal{O}_{\mathbb{P}^2}(d) \arrow[dl, "\beta"'] \arrow[d, dashed, "\alpha"] & \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^2} \arrow[r] & \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3} \arrow[r] & T_{\mathbb{P}^2} \arrow[r] & 0
\end{tikzcd}$$

apply Hom

$$0 \to \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}(d), \mathcal{O}_{\mathbb{P}^2}) \to \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}(d), \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3})$$

$$\begin{tikzcd} {} \arrow[r] & \operatorname{Hom}(\mathcal{O}_{\mathbb{P}^2}(d), T_{\mathbb{P}^2}) \arrow[r] & \operatorname{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(d), \mathcal{O}_{\mathbb{P}^2}) \\ & \alpha \arrow[u, phantom, sloped, "\in"] & H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-d)) = 0. \arrow[u, phantom, sloped, "\cong"] \end{tikzcd}$$

so we get $\mathcal{O}_{\mathbb{P}^2}(d) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3}$

$$\Rightarrow d \leq 1 \qquad \Rightarrow \mu_H(K) \leq 1.$$

but $\mu_H(T_{\mathbb{P}^2}) = \frac{3}{2} > 1.$ ☐

For definition of sstability we intersect with H^{n-1}

if H is very ample, then $H^{n-1} \approx D_1 \cap \dots \cap D_{n-1} = C$ smooth curve

$D_i \in |H|$

and $\mu_H(\mathcal{F}) = \underbrace{\mu(\mathcal{F}|_C)}_{\text{slope on the curve.}}$

In order to understand stability on higher-dimensional varieties, can we restrict to curves? No!

Ex: if $\ell \subset \mathbb{P}^2$ line, then $T_{\mathbb{P}^2}|_\ell \cong \mathcal{O}(2) \oplus \mathcal{O}(1)$ not stable.

Theorem of Mehta-Ramanathan:

X normal proj. of dimension n, $H \to X$ ample

H-stable

If $\mathcal{F} \to X$ is H-sstable then for $m \gg 0$ (depends on X, H and $\mathcal{F}$) if $D_1, \dots, D_{n-1} \in |mH|$ are general and $C = D_1 \cap \dots \cap D_{n-1}$ then

stable

$\mathcal{F}|_C$ is sstable

We call C a MR-curve.

Remark: If $\mathcal{F}$ is not H-sstable, there exists a HN-filtration.

Tomorrow: positivity + relation with sstability.