Kodaira's classification of algebraic surfaces.
Enriques–

minimal surfaces = surface without (-1)-curves.

$\kappa(X)$: Kodaira dimension, $h^0(X, mK_X) \sim c \cdot m^{\kappa}$

$\kappa(X) = 2$: general type

$\kappa(X) = 1$: elliptic fibration

$\kappa(X) = 0$ $\longrightarrow$
- K3 : $K_X \sim 0$, $h^1(\mathcal{O}_X) = 0$
- abelian : $K_X \sim 0$, $h^1(\mathcal{O}_X) = 2$
- Enriques : $2K_X \sim 0$, $K_X \not\sim 0$, $h^1(\mathcal{O}_X) = 0$
- bi-elliptic : $12K_X \sim 0$, $h^1(\mathcal{O}_X) = 1$.

$\kappa(X) = -\infty$: ruled surface

Example of K3:

$X_4 \subseteq \mathbb{P}^3$, $X_2 \cap X_3 \subseteq \mathbb{P}^4$, $X_2 \cap X_2 \cap X_2 \subseteq \mathbb{P}^5$

$\boxed{\text{Ex}}$ Show that these are the only possible K3 c.i $\subseteq \mathbb{P}^n$.

Kummer type A: abelian surface.

$\tau : A \longrightarrow A$ involution. 16 fixed points. $16 = 2^4$
$a \longmapsto -a$

$A/\tau \longleftarrow \widetilde{A/\tau}$

A/τ: 16 A_1-singularities; $\widetilde{A/\tau}$: smooth K3.

Now suppose
- X : K3.
- L : nef & big divisor

- nef $\overset{\text{def}}{\Longleftrightarrow}$ $(L.C) \geq 0$ $\forall$ $C \subseteq X$ curve
- big $\overset{\text{def}}{\Longleftrightarrow}$ $h^0(X, mL) \sim c \cdot m^2$ $\underset{\text{+nef}}{\Longleftrightarrow}$ $(L^2) > 0$

Fact 1 (Kodaira–Ramanujan vanishing)

X : proj sm surface.
L : nef & big
$\Rightarrow$ $H^i(X, K_X + L) = 0$ $\forall i > 0$

$\left(\overset{\shortparallel}{L}\right)$ K3

Fact 2 (R-R)

$\chi(X, L) = \frac{1}{2}L(L-K_X) + \chi(\mathcal{O}_X)$

$\underset{K3}{\rightsquigarrow}$ $\chi(X, L) = \frac{1}{2}L^2 + 2$

Thm (Saint-Donat, Mayer)

X: K3. L: nef & big

$\Rightarrow$ (1) if L not free, then $L = mE + C$

where $m \geq 2$, E: elliptic, $C \cong \mathbb{P}^1$, $E.C = 1$, $Bs|L| = C$

$m = h^0(L) - 1$

(2) $2L$: free,

Pf: (1) Write $L = M + F$ 　 F: fixed part, M: movable part.

Case 1 $F \neq 0$

claim 1 $\forall$ curve $C \subseteq F$, $C \cong \mathbb{P}^1$ 　　 claim 1' $\forall_{0<} F' \leq F$, $(F')^2 < 0$

$\because$ $h^0(C) = 1$

$h^0(C) - h^1(C) + \cancel{h^2(C)} = \frac{1}{2}C^2 + 2$

$\Rightarrow C^2 < 0$

$C^2 = (K_X + C).C = 2p_a(C) - 2 \geq -2$

$\Rightarrow p_a(C) = 0 \Rightarrow C \cong \mathbb{P}^1$.

Claim 2 $\forall_{0 \leq} F' < F$, $\overset{M'}{M + F'}$ is not $\boxed{\text{nef \& big}}$

$\because$ if so, $\Rightarrow h^0(M') = h^0(L) \Rightarrow \chi(M') = \chi(L)$

$\Rightarrow M'^2 = L^2$

$\Rightarrow \underset{(F-F')}{\underset{\parallel}{(L-M')}}\underset{(2M'+F-F')}{\underset{\parallel}{(L+M')}} = 0$ 　 $\Rightarrow (F-F')^2 = 0$

$\Rightarrow \chi(F-F') = 2$

$\underset{h^0 - h^1 + \cancel{h^2}}{\parallel}$ $\Rightarrow h^0(F-F') \geq 2$ $\lightning$

Claim 2 ($F' = 0$) $\Rightarrow$ M : not $\boxed{\text{nef}}$ & big

$\Rightarrow M^2 = 0$

Claim 3 $M = mE$: free. E : elliptic curve $m = h^0(M) - 1 \geq 2$.

$$m = h^0(M)-1 = h^0(L)-1 = \tfrac{1}{2}C^2+1$$

$\because$ M : movable $+ M^2 = 0 \Rightarrow M$: free

$$\begin{tikzcd} X \arrow[r, "\Phi_M"] \arrow[d, "f"'] & \mathbb{P}^m \\ Z \arrow[ur] & \end{tikzcd}$$

$M = f^* H_Z$.

f: connected fiber

- $M^2 = 0 \Rightarrow \dim Z = 1 \Rightarrow Z \cong \mathbb{P}^1$
  $h^1(\mathcal{O}_X) = 0 \geq h^1(\mathcal{O}_Z)$
- $h^0(M) = h^0(H_Z) = m+1 \Rightarrow \deg H_Z = mP$.
- $E = f^*P$ general fiber $K_X \sim 0 \Rightarrow K_E \sim 0 \Rightarrow E$: elliptic.

we showed that $L = mE + F$. $F = \sum_{i=1}^r a_i C_i$

$0 < L^2 = (mE+F)^2 = 2mE.F + F^2 < 2m \cdot E \cdot F$

$\Rightarrow \exists\, C \subseteq F, (E.C) \geq 1$

$\Rightarrow (mE+C)^2 = 2mE.C + C^2 > 0 \Rightarrow mE+C$: big

$(mE+C).C \geq 0 \Rightarrow mE+C$: nef

Claim 2 ($F' = C$) $\Rightarrow C = F \Rightarrow L = mE + C$. (Ex $(E.C) = 1$)

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(2) $2L$: not free $\Rightarrow 2L = mE + C$

$\Rightarrow 2L.E = C.E = 1$ $\lightning$

Case 2 $F = 0 \Rightarrow L = M$: movable $\xRightarrow[\text{Claim}]{}$ L : free.

Claim 1: a general member of $|M|$ is irreducible

((Bertini + Zariski) $M \sim C_1 + C_2 + \cdots + C_r \sim rC_1$ $C_i \sim C_j$

$\Rightarrow h^0(M) = r+1,\ h^0(C_i) = 2.$ $\boxed{M^2 = r^2 C_1^2}$

$$\begin{cases} \frac{1}{2}r^2c^2+2=r+1 \\ \frac{1}{2}c^2+2=2 \end{cases} \quad \lightning$$

Claim 2 M: free.

Def: D: divisor on X

D: m-connect $\overset{\text{def}}{\iff}$ $\forall D=D_1+D_2$ ($D_1, D_2 \neq 0$, ≥ 0), $(D_1 \cdot D_2) \geq m$

Ramanujan's lemma: D: 1-connect $\Rightarrow H^1(X, \mathcal{O}_X(-D))=0$

Now we want $H^0(X, M) \twoheadrightarrow H^0(X, M\otimes k(x)) \quad \forall x \in X$

$\Leftarrow H^1(X, M\otimes m_x)=0$

$$\begin{array}{c} \| \\ H^1(\tilde{X}, \pi^*M\otimes \mathcal{O}_{\tilde{X}}(-E)) \\ \| \\ \omega_{\tilde{X}}\otimes \boxed{\pi^*M\otimes\mathcal{O}_{\tilde{X}}(-2E)} \end{array}$$

$\tilde{X} \to X$, $E \subset \tilde{X}$, $x \in X$, $E \to x$

$\omega_X \cong \mathcal{O}_X$

$\rightsquigarrow \omega_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(E)$

$\Leftarrow \pi^*M\otimes\mathcal{O}_{\tilde{X}}(-2E)$: 1-connect.

$\overset{\text{omit.}}{\Longleftarrow}$ $|M|$: 2-connect (Recall M: irreducible reduced).

Pf: $|M|$: 1-connect. ✓

Suppose $M \sim C_1+C_2 \quad C_1\cdot C_2=1$.

① if $M.C_i \geq 2 \Rightarrow (C_i^2) \geq 1 \Rightarrow (C_i^2)\geq 2 \overset{\text{HIT}}{\Rightarrow} (C_1.C_2)\geq 2 \quad \lightning$

② if $M.C_1 = 0 \Rightarrow C_1^2=-1 \quad \lightning$

③ if $M.C_1 = 1 \Rightarrow C_1^2 = 0 \overset{\text{RR}}{\Longrightarrow} h^0(C_1)\geq 2$

$0 \to \mathcal{O}_X(-C_2) \to \mathcal{O}_X(C_1) \to \mathcal{O}_M(C_1) \to 0$

$\Rightarrow h^0(M, \mathcal{O}_M(C_1)) \geq 2. \quad \deg_M C_1 = 1$

$\Rightarrow M \cong \mathbb{P}^1 \Rightarrow M^2 < 0$

Reider's theorem:

Thm (Reider) X: sm. proj surface L: nef divisor

if $L^2 \geq 5$ & $p \in Bs|K_X+L|$

then $\exists$ curve $E \ni p$ s.t.

$\begin{cases} (1)\ L.E=0,\ E^2=-1 & \text{or} \\ (2)\ L.E=1,\ E^2=0 \end{cases}$

Apply to K3 L: nef $L^2 \geq 2 \Rightarrow (2L)^2 \geq 8$

$(K=0)$ if $p \in Bs|2L| \Rightarrow \exists E \ni p,\ \begin{cases} \cancel{2L.E=0},\ \cancel{E^2=-1} & \times \\ \cancel{(2L.E)=1},\ E^2=0 & \times \end{cases}$

Definition X: variety. D: div. Z: 0-cycle $=\sum a_i P_i$

Z: special position wrt $|D|$

$\overset{\Longleftrightarrow}{\text{def}} \begin{cases} (1)\ H^0(X, D) \to H^0(Z, D) \text{ not surj} \\ (2)\ H^0(X, D \otimes I_{Z'}) = H^0(X, D \otimes I_Z)\ \ \forall Z' \subseteq Z,\ \deg(Z-Z')=1 \end{cases}$

Thm (Griffith-Harris) X: surface.

TFAE: (1) Z : sp wrt $|K_X+L|$

$\Downarrow$

(2) $\exists (\mathcal{E}, e)$, $\mathcal{E}$: rk 2 bundle. e: section

s.t. $\det \mathcal{E} = \Lambda^2 \mathcal{E} = L$, $(e=0) = Z$.

Pf of Reider's thm $p \in Bs|K_X+L|$

$\Rightarrow p$: special

$\Rightarrow \exists (\mathcal{E}, e)$. $\begin{cases} (e=0) = p \\ \det \mathcal{E} = L \end{cases}$

$c_1^2(\mathcal{E}) = L^2$

$c_2(\mathcal{E}) = \deg Z = 1$

$\Rightarrow \ c_1^2(\mathcal{E}) > 4c_2(\mathcal{E}) \Longrightarrow$ Bogomolov <u>unstable</u>

$$\begin{tikzcd}
 & 0 & & \\
 & I_A \otimes \mathcal{O}_X(E) \arrow[u] & & \\
0 \arrow[r] & \mathcal{O}_X \arrow[r, hook, "e"] & \mathcal{E} \arrow[u] \arrow[r] & I_p \otimes \mathcal{O}_X(L) \arrow[r] & 0 \\
 & & \mathcal{O}_X(M) \arrow[u] \arrow[ur, dashed, "\text{no-zero}"'] & & \\
 & & 0 \arrow[u] & &
\end{tikzcd}$$

<u>Stability</u>: fix H: ample line bundle $\quad \mathcal{E}$: v.b

$$\mu_H(\mathcal{E}) := \frac{1}{\operatorname{rk}\mathcal{E}} \cdot (c_1(\mathcal{E}) . H)$$

$$\mathcal{E}: \text{semistable} \underset{\text{def}}{\Longleftrightarrow} \forall\ \mathcal{E}' \subseteq \mathcal{E},\ \mu_H(\mathcal{E}') \leq \mu(\mathcal{E})$$

unstable $\Rightarrow$ $0 \to \exists\mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$

$$\mu(\mathcal{E}') > \mu(\mathcal{E})$$

$$\mathcal{E}' = \mathcal{O}_X(M), \quad \mathcal{E}'' = I_A \otimes \mathcal{O}_X(E).$$

$\rightsquigarrow$
(a) $M + E = L$

(b) $M.E + \deg A = c_2(\mathcal{E}) = \deg p = 1.$

(c) $(M-E)^2 = L^2 - 4M.E \underset{+}{>} 0$

$(M-E).H > 0 \quad (\text{HIT} \Rightarrow \forall H\ ).$

$\mathcal{O}_X(M) \longrightarrow I_p \otimes \mathcal{O}_X(L)$ non-zero (otherwise, $\mathcal{O}_X(M) \hookrightarrow \mathcal{O}_X$

$\Rightarrow M \leq 0 \Rightarrow (M-E).H \overset{\parallel}{} (2M-L).H < 0$)

$\Rightarrow \ \mathcal{O}_X \longrightarrow I_p \otimes \mathcal{O}_X(\underset{\overset{\parallel}{E \geq 0}}{L-M})$

$\Rightarrow \ E \geq 0$ & $p \in E$.

$$\Rightarrow \begin{cases} \circ\ b) \Rightarrow L\cdot E - E^2 \leq 1 & ① \\ \circ\ L\cdot E \geq 0 & ② \\ \circ\ L^2 \cdot E^2 \leq (L\cdot E)^2 & ③ \\ \circ\ J \Rightarrow L^2 \geq 2(L\cdot E) & ④ \\ \circ\ L^2 \geq 5 & ⑤ \end{cases}$$

$$L\cdot E \leq 1 + E^2 \leq 1 + \frac{(L\cdot E)^2}{L^2} \leq 1 + \frac{(L\cdot E)}{2} \Rightarrow \underline{L\cdot E \leq 2}$$

$\circ = 2 \Rightarrow L^2\cdot E^2 = (L\cdot E)^2 = 4$ ξ.

$\Rightarrow (L\cdot E) \leq 1$. if $L\cdot E = 1 \Rightarrow E^2 = 0$ $\underline{①+③}$

$L\cdot E = 0 \Rightarrow E^2 = \begin{cases} \cancel{0} \\ -1 \end{cases}$ $\underline{\text{HIT}}$

Hyperkähler manifolds.

X: compact Kähler manifold $c_1 = 0$

$\uparrow \pi$: étale (Beauville-Bogomolov decomposition theorem)

$$\tilde{X} = \underbrace{A}_{\substack{\text{complex torus} \\ \text{(Abelian var)}}} \times \prod_i \underbrace{X_i}_{\text{Calabi-Yau}} \times \prod_j \underbrace{Y_j}_{\text{hyperkähler (HK)}}$$

X: CY $\overset{}{\underset{\text{def}}{\Leftrightarrow}} \begin{cases} X: \text{simply connected}, \ K_X \sim 0 \\ h^i(X, \mathcal{O}_X) = 0 \quad 0 < i < \dim X \end{cases}$

X: HK $\overset{\text{def}}{\Leftrightarrow} \begin{cases} X: \text{simply conn. } (K_X \sim 0) \\ H^0(X, \Omega^2_X) = \mathbb{C}\langle\sigma\rangle \quad \sigma: \text{every where non-degenerate 2-form.} \end{cases}$

dim $= 2n$

$(\rightsquigarrow H^0(X, \omega_X) = \mathbb{C}\langle \underline{\underline{\sigma^n}} \rangle \Rightarrow \omega_X \cong \mathcal{O}_X)$.

2. Day 2

Exercise 2.1. Show that general complete intersections

$$X_4 \subset \mathbb{P}^3,$$
$$X_{2,3} \subset \mathbb{P}^4,$$
$$X_{2,2,2} \subset \mathbb{P}^5$$

are K3 surfaces. Are there any other K3 surfaces coming from complete intersections in projective spaces?

Exercise 2.2. Let X be a smooth projective surface and $\pi : X' \to X$ be the blowup at a point with exceptional divisor E. Let D be a divisor on X such that every divisor in $|D|$ is 2-connected. Show that every divisor in $|\pi^* D - 2E|$ is 1-connected.

Exercise 2.3. Let D be a nef and big effective divisor on a K3 surface. Show that D is 1-connected.

Exercise 2.4. Let L be a line bundle on a (possibly singular) projective curve C. Suppost that $\deg L = 1$ and $h^0(C, L) \geq 2$. Show that $C \simeq \mathbb{P}^1$.