Positivity of the cotangent bundle

Talk II

Notation: X proj. variety, $V \to X$ vector bundle of rank r

$\pi : \mathbb{P}(V) \to X$ the projectivisation of V (sense of Grothendieck)

Set ξ_V = diviser associated to tautol. line bundle $\mathcal{O}_{\mathbb{P}(V)}(1)$

Useful properties:

1) If $V \to L \to 0$ quotient line bundle

$$\begin{tikzcd} \mathbb{P}(L) \subset \mathbb{P}(V) \arrow[d] \\ X \arrow[u, bend left] \end{tikzcd}$$

is a section of π and

$$\mathcal{O}_{\mathbb{P}(V)}(1)\big|_{\mathbb{P}(L)} \simeq L$$

2) More generally if $f: Y \to X$ morphism from proj. variety and

$f^* V \to L \to 0$ quotient line bundle, then

$\exists\, s: Y \to \mathbb{P}(V)$ s.t. $s^* \mathcal{O}_{\mathbb{P}(V)}(1) \cong L$.

Defn: A vector bundle $V \to X$ is nef / ample if the tautological diviser $\xi_V \to \mathbb{P}(V)$ is nef / ample.

Remark: If rk $V = 1$, then $\mathbb{P}(V) \simeq X$ and $\xi_V = c_1(L)$, so defn generalises defn for line bundle.

Ex: $V \to X$ nef vector bundle, $L \to X$ line bundle that is nef / ample

Then $V \otimes L$ is nef / ample.

Fact: if $V \to X$ nef / ample vector bundle, then $\det V$ is also nef / ample.

Prop (Kleiman criterion)

A vector bundle $V \to X$ is nef if and only if $\forall$ finite map $f : C \to X$ from a smooth projective curve C and every quotient line bundle $f^* V \to L \to 0$ one has $\deg L \geq 0$.

For the proof just recall that the tautological bundle is a quotient $\pi : \mathbb{P}(V) \to X$

$$\pi^* V \to \mathcal{O}_{\mathbb{P}(V)}(1) \to 0.$$

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Thus if $C' \subset \mathbb{P}(V)$ curve s.t. $\xi_V \cdot C' < 0$ then take normalisation $\nu : C \to C'$ this gives finite morphism $f = \pi \circ \nu : C \to X$ and

$$\begin{tikzcd}
f^*V \arrow[r] \arrow[d, equal] & \underbrace{v^*\mathcal{O}_{\mathbb{P}(V)}(1)}_{\text{line bundle of degree } \xi_V.C'<0.} \arrow[r] & 0 \\
v^*\pi^*V & &
\end{tikzcd}$$

Hartshorne's theorem:

C a smooth projective curve and $V \to C$ a vector bundle. Then

V is nef/ample $\iff$ $\forall$ quotient vector bundle

$$V \to Q \to 0$$

one has $\deg Q \overset{>}{\geq} 0$.

Important remark: we also have to check the "obvious quotient"

$$V \to V \to 0.$$

Mumford's example: $\exists$ vector bundle $V \to C$ of rank two s.t. $\deg V = 0$ and $\xi_V . C' > 0$ for all curves $C' \subset \mathbb{P}(V)$

Obviously V is nef, but is it ample?

If $V \to Q \to 0$ of $\operatorname{rk} Q = 1$, then

$$\deg Q = \xi_V . \mathbb{P}(Q) > 0$$

$V \to V \to 0$ has $\deg V = 0$.

Indeed ξ_V is not ample; Proof: Grothendieck relation

$$\xi_V^2 - \pi^* \underset{\underset{\displaystyle 0}{\shortparallel}}{c_1(V)}\, \xi_V = 0. \quad \Rightarrow \xi_V^2 = 0.$$

hence not ample.

$\mathbb{Q}$-twists

Recall: X proj. variety, $V \to X$ vector bundle, $L \to X$ line bundle. Then

$$\mathbb{P}(V \otimes L) \simeq \mathbb{P}(V) \text{ as a variety}$$

and $\underline{\xi_{V\otimes L} = \xi_V + \pi^* c_1(L)}$

Defn: X a projective variety. A $\mathbb{Q}$-twisted vector bundle is an ordered couple (V, δ) where

$V \to X$ is a vector bundle

and $\delta \in \underset{\underset{\displaystyle NS(X) \otimes \mathbb{Q}}{\wr\shortparallel}}{N^1(X)_{\mathbb{Q}}}$ is a $\mathbb{Q}$-divisor class we say that the $\mathbb{Q}$-twist (V, δ) is nef/ample if the class $\xi_V + \pi^*\delta$ on $\mathbb{P}(V) \to X$ is nef/ample.

Remark: for simplicity we will $V \otimes \delta$ instead of (V, δ), although this has no geometric sense.

Fundamental lemma (Miyaoka '80s)

Let C be a smooth proj. curve and $V \to C$ a vector bundle. Then $r = \operatorname{rank} V$

$$V \text{ is sstable} \iff V \otimes \tfrac{1}{r} \det V^* \text{ is nef.}$$

Have to admit some facts:

i) Kleiman's criterion true for $\mathbb{Q}$-twists

ii) V sstable $\iff$ $V \otimes \frac{1}{r} \det V^*$ sstable

iii) If $C' \to C$ finite cover of smooth proj. curves Then V sstable $\iff$ f^*V sstable.

Proof of Miyaoka's lemma:

By ii) we can work with $V \otimes \frac{1}{r} \det V^*$ instead of V.

But $\deg(V \otimes \frac{1}{r} \det V^*) =$

$$= \deg V + r \cdot \frac{1}{r} \deg \det V^* = 0.$$

Without loss of generality $\det V \equiv 0$.

- If V nef, let $K \subset V$ be a subbundle, so

$$V \to V/K \to 0 \quad \text{quotient bundle}$$

V nef $\overset{\text{ex.}}{\Rightarrow}$ V/K is also nef $\Rightarrow \det(V/K)$ nef, hence $\deg V/K \geq 0$.

$\overset{\deg V = 0}{\Rightarrow} \deg K \leq 0$

In particular $\mu(K) \leq 0 = \mu(V)$.

- Vice versa, assume V not nef. By Kleiman $\exists\, C' \to C$ finite map and

$$f^* V \to L \to 0 \quad \text{quotient line bundle}$$

with $\deg L < 0$.

but $\deg f^* V = 0$ since $\deg V = 0$.

$\Rightarrow f^*V$ is not sstable.

iii) $\Rightarrow$ V not sstable $\blacksquare$

Recall Hartshorne C curve, $V \to C$ vb.

V nef/ample $\Leftrightarrow$ $\forall$ quotient $V \to Q \to 0$ we have $\deg Q \overset{>}{\geq} 0$.

Proof of Hartshorne's thm:

Assume $\forall$ $V \to Q \to 0$ we have $\deg Q \overset{>}{\geq} 0$

we want to show that V is $\overset{\text{ample}}{\text{nef}}$.

Step 1: Assume V is semistable.

Miyaoka $\Leftrightarrow$ $V \otimes \frac{1}{r} \det V^*$ is nef

The obvious quotient $V \to V \to 0$ gives $\deg V \overset{>}{\geq} 0$

we see that

$$\boxed{V \simeq \underbrace{\left(V \otimes \tfrac{1}{r} \det V^*\right)}_{\text{nef}} \otimes \underbrace{\tfrac{1}{r} \det V}}$$ is ~~nef~~ ample

~~nef~~ since its degree $\overset{>}{\geq} 0$.

ample

Step 2: General case.

Look at HN-filtration of V

$0 \subset HN_1(V) \subset \dots \subset HN_{\ell-1}(V) \subset V$. Then

$$\mu(Gr_1(V)) > \dots > \underset{\substack{\| \\ V/HN_{\ell-1}(V)}}{\mu(Gr_\ell(V))} \geq 0.$$

quotient of V, hence by assumption deg ≥ 0.

$\Rightarrow$ All graded pieces of HN-filtration are nef vector bundles (Proof as in Step 1)

General fact: if

$0 \to K \to V \to Q \to 0$ extension

of vector bundles s.t. K and Q are nef,

then V is nef.

$\Rightarrow V$ is nef $\blacksquare$

Hartshorne's thm is false for every singular curve:

If C is singular proj. curve, let

$\nu : \tilde{C} \to C$ be its normalisation.

if $p \in C$ is a smooth point, there exists an unsplit extension $V \not\simeq \mathcal{O}_C \oplus \mathcal{O}_C(p)$

$$0 \to \mathcal{O}_C \to V \to \mathcal{O}_C(p) \to 0$$

s.t. $\nu^* V \simeq \mathcal{O}_C \oplus \mathcal{O}_C(p)$.

Then $\deg V > 1$ and if $V \to Q \to 0$ quotient line bundle, then

$$\deg Q > 0.$$

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_C \arrow[r] \arrow[dr, "\alpha"'] & V \arrow[d] \\
& & Q
\end{tikzcd}$$

if $\alpha \neq 0$ then $\deg \overset{0}{\overset{\shortparallel}{\mathcal{O}_C}} \leq \deg Q$

and $=$ iff α isom (excluded since then $V \simeq \mathcal{O}_C \oplus \mathcal{O}_C(p)$)

Nevertheless V is not ample, since

$\nu^* V \simeq \mathcal{O}_C \oplus \mathcal{O}_C(p)$ is not ample. ∎

Defn: X projective variety, $V \to X$ vector bundle (or $\mathbb{Q}$-twisted vector bundle)

We say that V is pseudoeffective / big if the tautological divisor $\xi_V \to \mathbb{P}(V)$ is pseff/big.

Ex: If $V \to X$ vector bundle s.t. $H^0(X, S^m V) \neq 0$ for some $m \in \mathbb{N}$.

Then V is pseff.

Proof:
$$0 \neq H^0(X, \underset{\pi_* \mathcal{O}_{\mathbb{P}(V)}(m)}{\underset{\parallel}{S^m V}}) \simeq H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(m))$$

$\Rightarrow \mathcal{O}_{\mathbb{P}(V)}(m)$ effective

$\Rightarrow \mathcal{O}_{\mathbb{P}(V)}(1)$ pseudoeffective. $\blacksquare$

For line bundle we know that pseff = limit of effective $\mathbb{Q}$-divisors.

More precisely: fix $A \to X$ ample line bundle, then

$L \to X$ is pseff $\Longleftrightarrow$ $\forall\, \varepsilon > 0$ there exist $i, j \in \mathbb{N}$ s.t. $\frac{j}{i} < \varepsilon$ and $H^0(X, L^{\otimes i} \otimes A^{\otimes j}) \neq 0$.

NB: $\frac{1}{i} \underbrace{c_1(L^{\otimes i} \otimes A^{\otimes j})}_{\text{represented by effective divisor}} = c_1(L) + \underbrace{\frac{j}{i} c_1(A)}_{\text{very small.}}$

Lemma (Druel) Fix $A \to X$ ample.

A vector bundle $V \to X$ is pseff iff

$\forall \varepsilon > 0$ there exist $i, j \in \mathbb{N}$ s.t. $\frac{j}{i} < \varepsilon$ and

$$H^0(X, S^i V \otimes A^{\otimes j}) \neq 0.$$

For proof, be careful: $\pi^* A$ is not ample, so criterion for line bundles does not apply.

By Serre's theorem $\exists\, m_0 \in \mathbb{N}$ s.t.

$$V \otimes A^{\otimes m_0} \text{ is ample.}$$

$\overset{\text{defn}}{\Longleftrightarrow}$ $\boxed{\underline{\mathcal{O}_{\mathbb{P}(V)}(1) \otimes \pi^* A^{\otimes m_0}} \text{ is ample.}}$

Thus if $\boxed{\xi_V \text{ is pseff,}}$ then for all $m \in \mathbb{N}$ the bundle

$$\mathcal{O}_{\mathbb{P}(V)}(m) \otimes \mathcal{O}_{\mathbb{P}(V)}(1) \otimes \pi^* A^{\otimes m_0} \text{ is big.}$$

$$\Rightarrow \exists\, k_m \in \mathbb{N} \text{ s.t. } H^0\left(\mathbb{P}(V), \left(\mathcal{O}_{\mathbb{P}(V)}(m+1) \otimes \pi^* A^{\otimes m_0}\right)^{\otimes k_m}\right) \neq 0$$

$$\cong$$

$$H^0\left(X, S^{k_m (m+1)} V \otimes A^{k_m m_0}\right)$$

$$\frac{k_m m_0}{k_m (m+1)} = \frac{m_0}{m+1} \longrightarrow 0 \quad \text{for } m \to \infty.$$

$\square$

Warning: nef vector bundles are nice,

psef vector bundles are horrible.

Examples:

1) If $V \to X$ nef, then $\det V$ is nef.

If $V \to X$ psef, then $\det V$ can be antiample.

Ex: $V \cong A \oplus (A^*)^{\otimes 2}$ A ample.

$\det V \cong A^*$

2) $V \to X$ nef $\iff$ $S^m V$ nef for some $m \in \mathbb{N}$.

„$\Leftarrow$" is not true for pseudoeffectivity

X K3 surface (X smooth proj, $K_X \cong \mathcal{O}_X$, $\pi_1(X) = 1$)

Then $V = \underline{\Omega_X}$ is not pseudoeffective,
(proof on Thursday)

but $S^2 V$ is pseudoeffective.

Proof: $$S^2(S^2V) = S^4 V \oplus \underset{\mathcal{O}_X^{\otimes 2}}{\underset{\parallel}{\det V^{\otimes 2}}}$$

$$\Rightarrow H^0(X, S^2(S^2V)) \neq 0.$$

$$\overset{ex}{\Rightarrow} S^2 V \text{ is pseff. } \blacksquare$$

Challenge: given a vector bundle $V \to X$ that is pseudoeffective, but not necessarily nef, can we say something about the non-nef locus?

Next talk: why is this an important challenge
+ some results if $c_1(V) = 0$.

V vector bundle is big iff
$\exists$ ample $\mathbb{Q}$-divisor A s.t. $V \otimes A^*$ is pseff.

V big iff $\exists$ s.t.

$$H^0(X, S^m(V \otimes A^*)) \neq 0$$

for some $m \in \mathbb{N}$.