(II) Fibration case:

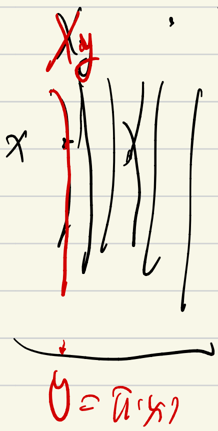
Thm: $\pi: X \longrightarrow Y$ be a proper fibration between two Kähler mfds.

$L \to X$ holo line bundle. $i\Theta_{h_L}(L) \geq 0$ h_L C^∞ herm.

If $\pi_*(K_{X/Y}+L) \neq 0 \Rightarrow$

- $\exists\, h_B$ $K_{X/Y}+L$ s.t. $i\Theta_{h_B}(K_{X/Y}+L) \geq 0$ on X
- $\forall y \in Y$ generic pt, $\forall s \in H^0(X_y, K_{X/Y}+L)$, $\Rightarrow$ $|s|^2_{h_B} \leq C$ on X_y

$\uparrow$ L^∞



Zariski open set.

proof: Let $Y^0 \subset Y$ be the locus s.t. π is a submersion over Y^0.

Let $X^0 = \pi^{-1}(Y^0)$.

$\pi: X^0 \longrightarrow Y^0$ is submersion

By Thm 1: $\exists\, h_B$ on $K_{X/Y}+L\,|_{X_0}$, s.t. $i\Theta_{h_B}(K_{X/Y}+L) \geq 0$ on X^0.

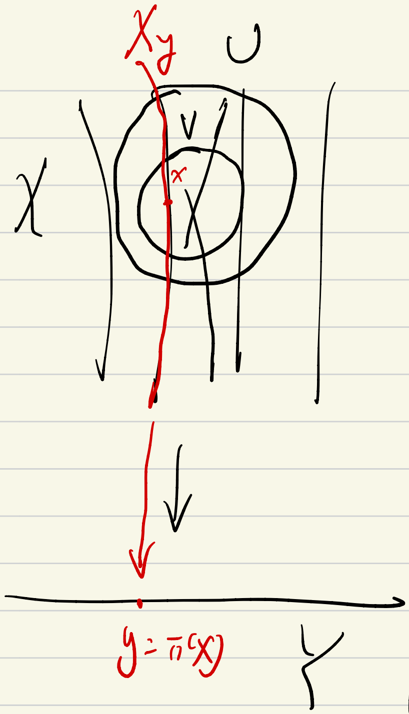
- $x \in X^0$. $\xi \in (K_{X/Y}+L)_x$

$$|\xi|^2_{h_B}(x) := \boxed{\sup_{\substack{s \in H^0(X_y, K_{X/Y}+L) \\ \int_{X_y} |s|^2_{h_L} = 1}} \frac{|\xi(x)|^2}{|s(x)|^2}}$$

We want to extend h_B on $(K_{X/Y}+L)|_{X_0}$ to total space $K_{X/Y}+L$ on X.

and $i\Theta_{h_B} \geq 0$ on X

By Rieman extension, it is suffient to prove,



Let $U \subset X$ a small topo open set.

$V \subset\subset U$ open subset of compact support in U.

Let σ be a holo base of $K_{X/Y}+L|_U$.

$$|\sigma|^2_{h_B} = e^{-\varphi} \text{ on } U\cap X_0 \qquad i\Theta_{h_B} = i\partial\bar\partial\varphi \geq 0$$

$$\boxed{\text{Aim: } \varphi \leq C \text{ on } V\cap X_0 \quad (*)}$$

If (*) is proved, by Riemann extension, φ can be extend as a psh function on V, $i\Theta_{h_B} = i\partial\bar\partial\varphi \geq 0$ on V.

proof of (*): $\boxed{\text{Let } x \in V\cap X_0}$. By the construction of h_B

$$\exists\, s \in H^0(X_y, K_{X_y}+L), \text{ s.t. } \int_{X_y} |s|^2_{h_L} = 1, \quad \boxed{|\sigma|^2_{h_B}(x) = \frac{|\sigma(x)|^2}{|s(x)|^2}}$$

Since U is a ~~small~~ small open set, we can suppose that

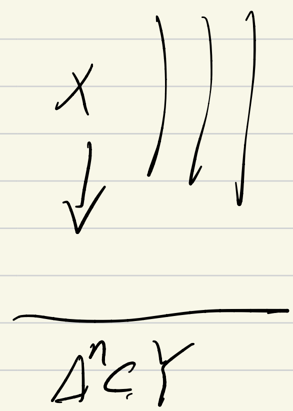
$$\boxed{\pi(U)} \subset \boxed{\Delta^n} \subset Y^n, \quad dz = dz_1\wedge\cdots\wedge dz_n$$

$\boxed{\text{By O-T thm}}$, we can extend s to $\boxed{\pi^{-1}(\Delta^n)}$, i.e.,

$$\exists\, \tilde{s} \in H^0(\pi^{-1}(\Delta^n), K_X+L) \text{ s.t.}$$

$$\int_U |\tilde{s}|^2_{h_L} \leq \int_{\pi^{-1}(\Delta^n)} |\tilde{s}|^2_{h_L} \leq C(U)\int_{X_y} |s|^2_{h_L} = C(U), \qquad \tilde{s}|_{X_y} = s\wedge \pi^* dz$$

$\tilde{s}|_U \in H^0(U, K_X+L)$ $\qquad$ σ base of $K_{X_Y}+L$ on U.



$\Rightarrow \tilde{s} = \underset{\substack{\uparrow \\ \text{holo function}}}{f} \cdot (\sigma \wedge \pi^* dz)$

$\Rightarrow \tilde{s}|_{X_y} = s \wedge \pi^* dz$

$\Rightarrow \boxed{f \cdot \sigma \wedge \pi^* dz|_{X_y} = s \wedge \pi^* dz}$

$\boxed{\int_U |\tilde{s}|^2_{h_L} \le C(U)}$

Since h_L $C^\infty \Rightarrow h_L \ge c > 0 \Rightarrow \boxed{\int_U |f|^2 \le C'(U)}$

mean value ineq

$\Rightarrow \sup_{z \in V} |f|(z) \le C''(U, V)$

$\Rightarrow |f|(x) \le C''(U, V)$

On the other hand, $\boxed{|\sigma|^2_{h_B}(x)} = \left|\frac{\sigma(x)}{s(x)}\right|^2 = \frac{1}{|f(x)|^2}$

$e^{-\varphi}$ //

$\forall x \in V \cap X_0$.

$\Rightarrow \varphi(x) = \ln |f(x)|^2 \le \ln C''(U, V)$

$\Rightarrow \varphi$ is upper bonded on V $\square$

in many cases:

Rk (1) For application: $i\Theta_{h_L}(L)\geq 0$ $\boxed{h_L \text{ may be singular.}}$

· If $\pi_*(K_{X/Y}\otimes L\otimes \mathcal{I}(h_L))\neq 0 \Rightarrow K_{X/Y}+L$ is psef. □

is constructed by

$$h_B: \left\{ s\in H^0(X_y, K_{X_y}+L), \ \boxed{\int_{X_y}|s|^2_{h_L}=1} \right\}$$

$$\Updownarrow$$

$$s\in H^0(X_y, K_{X_y}\otimes L\otimes \mathcal{I}(h_L|_{X_y}))$$

(2): If $\pi_*(K_{X/Y}\otimes L\otimes \mathcal{I}(h_L))=0$

We may consider $\pi_*(mK_{X/Y}\otimes L\otimes \mathcal{I}_m(h_L))$

$m\in\mathbb{N}^*$. $\mathcal{I}_m(h_L):=\left\{ f\in \mathcal{O}_{X,x} \ \middle|\ \int_{U_x}(|f|_{h_L})^{2/m}<+\infty \right\}$

Thm 3 $\boxed{\text{If } \pi_*(mK_{X/Y}\otimes L\otimes \mathcal{I}_m(h_L))\neq 0 \Rightarrow mK_{X/Y}+L \text{ is psef.}}$

Idea: We use the $L^{2/m}$-optimal O-T:

Thm (O-T). $\pi: X\longrightarrow \Delta^n$ proper fibration X Kähler mfld.

We suppose: $X_0=\pi^{-1}(0)$ is smooth, $L\to X$ $i\Theta_{h_L}(L)\geq 0$

Let $s\in H^0(X_0, mK_{X_0}+L)$. $\boxed{\text{If } \int_{X_0}|s|^{2/m}_{h_L}<+\infty}$ and

$\exists$ U nb of $0\in\Delta^n$, s.t. $\exists\ \tilde{s}\in H^0(\pi^{-1}(U), mK_{X/Y}+L)$

$\tilde{s}|_{X_0}=s$, $\int_{\pi^{-1}(U)}|\tilde{s}|^{2/m}_{h_L}\, dz\wedge d\bar{z}<+\infty$

$\Rightarrow \exists\ \tilde{S} \in H^0(X, mK_{X/Y} + L)$,

- $\int_X |\tilde{S}|_{h_L}^{2/m} \pi^*(dz \wedge d\bar{z}) \leq \mathrm{vol}(\Delta^n) \cdot \int_{X_0} |S|_{h_L}^{2/m}$.
- $\tilde{S}|_{X_0} = S$.

$\square$

- Thm (O-T) + Thm 1 + Thm $\Rightarrow \boxed{\text{Thm } 3}$. $\square$

<u>Indication</u>: $\underline{\mathcal{I}(mK_{X/Y}+L)}_x$. $\boxed{|\tau|^2_{h_B}(x) := \sup\limits_{\substack{S \in H^0(X_y, mK_{X/Y}+L) \\ \int_{X_0} |S|_{h_L}^{2/m} = 1}} \frac{|\tau(x)|^2}{|S(x)|^2}}$

<u>Rk</u>:

- If $\exists$ $\underline{A \text{ ample on } X}$ $\Rightarrow$ $\circledast$ holds $\boxed{\text{Invariance of plurigenera}}$

Siu

X Kähler conj: $\circledast$ holds ?

$\square$

- <u>Singular metric on vector bundles</u>:

<u>Motivation</u>: - <u>Thm 2</u>: $\pi : X \to Y$ proper submersion

$(\pi_*(K_{X/Y}+L), h)$ $\quad i\Theta_h \geq 0$ Griffiths

C^∞ herm.

We want to generalize Thm 2 to $\underline{\pi \text{ is fibration}}$.
We should consider $\underline{\text{sig metrics}}$ on $\underline{\text{vector bundles}}$.

Def: Let $V \to X$ holo vector bundle. X complex mfd.

We say that h is a possible singular herm metric on V, if

- $\forall x \in X$: $h_x : V_x \to [0, +\infty]$ is a herm form

i.e., $\forall u, v \in V_x$

$\|u\|_{h_x} \in [0, +\infty]$.

$\lambda \in \mathbb{C}$ $\cdot\ \|\lambda u\|_{h_x} = |\lambda| \cdot \|u\|_{h_x}$

$\cdot\ \|u+v\|^2_{h_x} + \|u-v\|^2_{h_x} = 2(\|u\|^2_{h_x} + \|v\|^2_{h_x})$.

$+\infty = +\infty$

- $U \subset X$ open, $\forall s \in H^0(U, V)$, $\|s\|_h$ is measurable function on U.
- If $x \in X$ generic pt, $h_x : V_x \to]0, +\infty[$.

i.e outside a zero measure set

Def: We say $(V, h) \geq 0$ Griffith semipositive, if (h possible sing)

$\forall U \subset X$ topo open set $\forall s \in H^0(U, V^*)$, $\ln |s|^2_{h^*}$ is psh on U.

Rk: (1) $\boxed{h\ C^\infty \text{ herm}}$: $i\Theta_h(V) \geq 0$ Griffith $\Leftrightarrow$ $\boxed{\ln |s|^2_{h^*} \text{ psh}}$

h singular: $i\Theta_h(V)$ is not well-defined even in the sense of distribution.

(2) $\boxed{(V, h) \geq_{\mathcal{C}} 0 \text{. Griffith}} \Rightarrow \boxed{(\det V, \det h)}$

$i\Theta_{\det h}(\det V) \geq 0$ in the sense of current.

↑ well-defined in the sense of currents.

(3) (Rieman extension) $V \to X$ holo vector bundle.

Ex: $Y \subset X$ subvariety. $(V, h)|_{X \setminus Y} \geq_{\mathcal{C}} 0$ Griffith

① If $\operatorname{cod}_X Y \geq 2$: we can extend h to V on X.

s.t. $(V, h) \geq_{\mathcal{C}} 0$ on X.

② If $\operatorname{cod}_X Y = 1$: and $h \geq c \cdot Id$ on $X \setminus Y$ for some $c > 0$.

$\Rightarrow$ we can extend h to V on X.

s.t. $(V, h) \geq_{\mathcal{C}} 0$ on X.

(4) Let E_1, E_2 be two holo vector bundles on X.

and let $\varphi: E_1 \to E_2$ be a holo morphism.

- If φ is surjective on a generic pt $x \in X$. $(E_1)_x \to (E_2)_x$
- Let h_1 be a possible singular metric on E_1, $\boxed{(E_1, h_1) \geq 0}$.

$\Rightarrow h_1$ induces a --- h_2 on E_2, $\boxed{(E_2, h_2) \geq 0}$

quotient metric.

Let h_2 be quotient metric of h_1. $\qquad (E_1, h_1) \geq_{\mathcal{L}} 0$

$\varphi^*: (E_2^*, \underline{h_2^*}) \longrightarrow (E_1^*, h_1^*)$.

Let $s \in H^0(U, E_2^*)$.

$$\ln|s|_{h_2^*} = \ln|\varphi^*(s)|_{h_1^*} \text{ is psh.}$$

$\Rightarrow \ln|s|_{h_2^*}$ is psh. $\Rightarrow (E_2, h_2) \geq_{\mathcal{L}} 0$ $\square$

<u>Thm 4</u>. $\pi: X \to Y$ be a proper fibration between two Kähler mfds. $L \to X$ holo line bundle. $i\Theta_{h_L}(L) \geq 0$ $\Theta_{h,X}$

Let $\underline{Y_0}$ be the locall free locus of $\pi_*(K_{X/Y} \otimes L \otimes \mathcal{I}(h_L))$

$\Rightarrow \exists\, h$ on $\pi_*(K_{X/Y} \otimes L \otimes \mathcal{I}(h_L))$ on Y_0 $\neq 0$

possible singular

s.t. $(\pi_*(K_{X/Y} \otimes L \otimes \mathcal{I}(h_L)), h) \geq_{\mathcal{L}} 0$ on Y_0.

<u>Idea</u>. Let $\boxed{Y_1} \subset Y$ be the locus where π is submersion.

O-T $\Rightarrow$ extend to Y_0.

<u>Cor</u>. $(2) \Rightarrow i\Theta_{\det h}(\det \pi_*(K_{X/Y} \otimes L \otimes \mathcal{I}(h))) \geq 0$ on Y_0

line bundle

and $\operatorname{cod}_Y(Y\setminus Y_0) \geq 2$,

$\Rightarrow$ $i\Theta_{\det h}(\det \pi_*(K_{X/Y}\otimes L\otimes \mathcal{I}(h_L))) \geq 0$ on Y.

Rieman extension

$\Rightarrow$ $\det \pi_*(K_{X/Y}\otimes L\otimes \mathcal{I}(h_L))$ is psef on X.

Cor: If X_y be a generic fiber. if $\boxed{\mathcal{I}(h_L|_{X_y}) = \mathcal{O}_{X_y}}$

$\Rightarrow$ $\varphi: \pi_*(K_{X/Y}\otimes L\otimes \mathcal{I}(h_L)) \longrightarrow \pi_*(K_{X/Y}\otimes L)$

is iso on a generic pt $y \in Y$.

(4) $\Rightarrow$ $(\pi_*(K_{X/Y}\otimes L), h) \succeq_{\text{G}} 0$. on Y^0, where

Y^0 is the locally free locus of $\pi_*(K_{X/Y}+L)$.

$\boxed{\text{Cor}}$: If $\pi_*(mK_{X/Y}\otimes L) \neq 0$, and $\boxed{\mathcal{I}_m(h_L|_{X_y}) = \mathcal{O}_{X_y}}$ for a generic fiber X_y

$m \in \mathbb{N}^*$

$\Rightarrow$ $(\pi_*(mK_{X/Y}\otimes L), h) \succeq_{\text{G}} 0$.

Idea: $\boxed{mK_{X/Y}+L} = K_{X/Y} + \underbrace{\boxed{\frac{m-1}{m}\cdot(mK_{X/Y}+L) + \frac{1}{m}L}}_{L_1}$

By Thm 3 $\begin{cases} \exists\, h_B : \boxed{i\Theta_{h_B}(mK_{X_y/Y}+L)\geq 0.} \\ \exists\, h_L : i\Theta_{h_L}(L)\geq 0 \end{cases}$

Then $\boxed{L_1 := \frac{m-1}{m}(mK_{X_y/Y}+L)+\frac{1}{m}L}$

$$\left(L_1,\ \underbrace{\frac{m-1}{m}h_B+\frac{1}{m}h_L}_{=:\,h_1}\right)$$

$\Rightarrow i\Theta_{h_1}(L_1)\geq 0.$

$$\boxed{\pi_*(K_{X_y/Y}+L_1) = \pi_*(mK_{X_y/Y}+L)}$$

Exo: $\pi_*(K_{X_y/Y}\otimes L_1\otimes \mathcal{I}(h_1)) \simeq \pi_*(K_{X_y/Y}\otimes L_1)$ $\leftarrow$ $\mathcal{I}_m(h_1)|_{X_y}=\mathcal{O}_{X_y}$

$\nwarrow$ $|s|^2_{h_B}\leq C$ $\forall s\in H^0(X_y, mK_{X_y/Y}+L)$

Thm 4 $\Rightarrow (\pi_*(K_{X_y/Y}+L_1), h)\succcurlyeq 0$

$$\|$$

$(\pi_*(mK_{X_y/Y}+L), h)\succcurlyeq 0$ $\square$

For example : If $L = \sum E_i$ effective divisor.

For $m >> 1$, $\frac{1}{m} \cdot L = \sum \frac{1}{m} E_i$ klt.

$h_L := e^{-\sum_{1/m} |s_{E_i}|^2}$

$\Downarrow$

$\mathcal{I}_m(h_L) = \mathcal{O}_X$.

$\Rightarrow$ If $\pi_*(mK_{X/Y} + L) \neq 0$

then $(\pi_*(mK_{X/Y} + L), h) \succeq 0$.

• Relation between Griffith semipositive and Viehweg semipositive

X proj mfd. $\mathcal{F} \to X$ torsion free coherent sheaf.

Prop $(\mathcal{F}, h) \succeq_G 0$ on X_0 $\Longrightarrow$ $\mathcal{F}$ is Viehweg semipositive

X_0 : torsion free locus of $\mathcal{F}$.

$\not\Longleftarrow$

i.e. fix an ample line bundle A_X on X,

$\forall a > 0, \exists b \geq 0$, s.t.

$H^0(X, \widehat{\mathrm{Sym}}^{ab}(\mathcal{F}) \otimes \mathcal{O}_X(b \cdot A_X))$

$\twoheadrightarrow \widehat{\mathrm{Sym}}^{ab}(\mathcal{F}) \otimes \mathcal{O}_X(b \cdot A_X))_x$

proof: $\mathcal{F}$ is a vector bundle for simplicity

$$\begin{array}{c} \mathcal{O}(1) \\ \downarrow \mathbb{P}(\mathcal{F}) \\ \mathbb{P}(\mathcal{F}) \xrightarrow{\pi} X \end{array}$$

$$(\mathcal{F}, h) \succsim 0 \Rightarrow (\pi^*\mathcal{F}, \pi^* h) \succsim 0$$

induced metric

$$(\pi^*\mathcal{F}, \pi^* h) \twoheadrightarrow (\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1), h_1)$$

Rk (4): $\underline{i\Theta_{h_1}(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)) \geq 0}$

For A_X sufficient ample on X.

O-T: $H^0(\mathbb{P}(\mathcal{F}), K_{\mathbb{P}(\mathcal{F})/X} \otimes \mathcal{O}(m) \otimes \pi^*(A_X))$ $\qquad // \quad \mathrm{Sym}^{m-r}\mathcal{F} \otimes \det \mathcal{F} \otimes A_X$

$$\twoheadrightarrow H^0(\mathbb{P}(\mathcal{F})_x, K_{\mathbb{P}(\mathcal{F})/X} \otimes \mathcal{O}(m) \otimes \pi^*(A_X))$$

$\Rightarrow$ $\mathcal{F}$ is Viehweg semipositive.

$X = T^1$ 1-torus.

$0 \to \mathcal{O}_X \to V \to \mathcal{O}_X \to 0$ $\quad V$ is a nontrivial extension of $\mathcal{O}_X$ ~~det~~ on X.

$\Rightarrow$ • V Viehweg semipositive

• V is not Griffith semipositive. $c_1(V) = 0$

if $\exists (V, h) \geq_G 0 \Rightarrow i\Theta_{\det h}(\det V) \geq 0$

$\Rightarrow i\Theta_{\det h}(\det V) \equiv 0$

$\Rightarrow$ h C^∞. $i\Theta_h(V) \equiv 0$

$\Rightarrow$ V is herm flat.

$\Rightarrow$ contradiction. $\square$