MMP for Kähler 3-folds

Introduction

In algebraic case, for two varieties X, Y

They are birationally equivalent

$$\Longleftrightarrow$$

$$\begin{tikzcd} X \arrow[rr, dashed, leftrightarrow, "\text{bir equi}"] & & Y \\ U \arrow[u, hook, "\text{Zariski open}"] \arrow[rr, "\simeq"] & & V \arrow[u, hook, "\text{Zar open}"'] \end{tikzcd}$$

$$U \neq \phi$$

We aim to classify all smooth projective varieties up to bir equiv

In this topic, an important approach is the Minimal Model Program (MMP)

MMP comes from classification of proj surfaces (~ 1900)

Give a smooth proj surface $X = X_0$.

Then $\exists$ finitely many morphisms

$$X = X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} X_2 \to \cdots \xrightarrow{\varphi_{n-1}} X_n = X_{min}$$

s.t. * all φ_i are birational maps
* all X_i are smooth proj surfaces
* X_{min} contains no (-1)-curve

This procedure is a MMP for X.

Furthermore, we have a classification for all minimal surfaces (X_{min})
(Enriques-Kodaira classification)
~1950

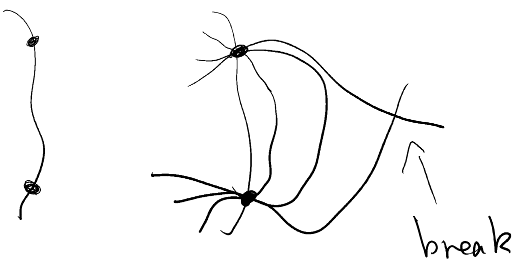
MMP for higher dimensional varieties.

Known result: - MMP for Proj 3-folds
- Classification for minimal 3-folds
(Abundence for proj 3-folds)
- In dim ≥ 4, we only have some partial result.
* MMP for Kähler 3-folds
(Höring-Peternell)

Classic Algebraic MMP

consists of several steps.

① Bend-and-break (find rational curves in a variety)



due to Mori

X proj smooth variety

② Cone Theorem: We have a discription for $\overline{NE}(X)$

$\overline{NE}(X)$: Mori cone, closed cone generated by curves in $N_1(X)$

$N_1(X)$: 1-cycle spaces modulo numerical equivalence

Thm: $\overline{NE}(X) = \overline{NE}(X)_{K_X \geq 0} + \sum_{i \in I} \mathbb{R}^+ [\Gamma_i]$

I is at most countable, Γ_i is a $K_X < 0$ curve.

③ Contraction Thm: $\forall i \in I$, $\exists$ contraction $c_i : X \longrightarrow Y$ s.t. a curve C is contracted by $c_i \Longleftrightarrow [C] \in \mathbb{R}^+ [\Gamma_i]$

④ Existence of flips. termination of MMP

[BCHM 10] ✓ Unknown in general

Kähler case

There are some difficulty.

1. B-and-B uses a technique, called "reduction modulo p"

   This technique does apply to Kähler varieties.

2. Cone theorem and contraction theorem.

   In alg case, contraction theorem uses linear system of certain nef divisors.

   In Kähler case, we might not able to find such a divisor.

Höring-Peternell's solution for Kähler 3-fold X

1. For B-and-B, they first use divisorial Zariski decomposition for K_X to obtain certain surfaces $\underbrace{\text{in}}_{S_1, \dots, S_r}$ in X

   Then we can use bend-and-break in these surfaces. and find rational curves in one of the surfaces $S_1, \dots, S_r$

2. For contraction thm, they use a theorem of Grauert.

Once we have these ~~two~~ theorems, the remainder part for MMP follows "almost" the same as in the algebraic case.

Further result: We have log MMP for klt pairs of Kähler 3-folds (Das-Ou, Das-Hacon)

<u>References</u>: Höring-Peternell / I. Minimal Model for Kähler 3-fol
\ II. Mori Fiber space for Kähler 3-folds.

Boucksom: Divisorial Zariski Decomposition on compact complex manifolds

Boucksom
Eyssidieux : An introduction to the Kähler-Ricci flow
Guedj Chapter 4, Section 4-6.

Kollár-Mori : Birational Geometry of algebraic varieties

Hartshorne.

I. Analytic varieties

<u>Def</u>: A holomorphic manifold is a topo space X with charts (U_i, φ_i), where $\{U_i\}$ is an open covering, and $\varphi_i : U_i \longrightarrow V_i \underset{\text{open}}{\subseteq} \mathbb{C}^n$ is a homeomorphism

s.t. the transition maps $\varphi_i \circ \varphi_j^{-1} : V_j \to V_i$ is biholomorphic

Def: (Analytic subvariety) Let $D \subseteq \mathbb{C}^N$ be an open connected domain. Let $X \subseteq D$ be a subset.

X is called an analytic subset if $\exists$ finitely many holomorphic functions $\{f_i\}$ on D such that X is the common zero locus of $\{f_i\}$.

More generally, Let $\mathcal{O}_D$ be the sheaf of holomorphic function on D. Then $(D, \mathcal{O}_D)$ is a locally ringed space

Let $f_1, \cdots, f_s$ be holomorphic functions on D. Then they define an analytic subset X, and an coherent ideal sheaf $\mathcal{I}$ on $(D, \mathcal{O}_D)$. Then

We get a locally ringed space $(X, \underset{\mathcal{O}_X}{\underset{\parallel}{\mathcal{O}_D/\mathcal{I}}})$

Then $(X, \mathcal{O}_X)$ is an analytic subvariety

Def: A complex analytic variety is a locally ringed space which is locally iso to analytic subvarieties defined as above.

Remark: We can define irreducible varieties, reduced varieties as for algebraic varieties.

Convention: * In this lecture, our varieties are reduced and irreducible. Hence we will omit the sheaf $\mathcal{O}_X$, and just write X for a variety.

* We use Euclidean (analytic) topology in this lecture.

Def: A function $f: X \to \mathbb{C}$ is a holomorphic function if $\exists$ covering $\{U_i\}$ of X such that

$$\text{each } U_i \xrightarrow{\sim} V_i \underset{\text{analy subvar}}{\subseteq} D_i \underset{\text{open domain}}{\subseteq} \mathbb{C}^{N_i}$$

and $f|_{U_i}$ comes from a holomorphic function on D_i.

Def: (meromophic) A "function" $f: X \dashrightarrow \mathbb{C}$ is meromorphic if it is locally the quotient of two holomophic functions.

Def: We generalize these notion to maps between analytic varieties.

Similarly, we define C^∞ functions $f: X \to \mathbb{R}$, or $g: X \to \mathbb{C}$.

We can define normal varieties as for alg varieties.
We also have resolution of singularities for complex varieties.

Rational singularities

Let $(X, \mathcal{O}_X)$ be a normal irreducible reduced variety.
Let $f: \hat{X} \to X$ be a resolution of sing
We say ~~that~~ X has rational singularities if the sheaves
$$R f_*^i \mathcal{O}_{\hat{X}} = 0 \quad \text{for all } i > 0$$

We define Weil divisors ~~in~~ the same way as for alg varieties: A weil divisor is a formal linear combination of subvarieties of codim 1.

Let X be a normal varieties, and D a Weil divisor on X. Then we can consider the sheaf
$$\mathcal{O}_X(D) = \text{mero functions whose pole } P \text{ satisfy } D - P \geq 0 .$$

This is a reflexive sheaf of rank one.

A Weil divison D is called $\mathbb{Q}$-Cartier if $\exists$ integer $m>0$ s.t. $\mathcal{O}_X(mD)$ is a locally free sheaf.

In this lecture, we set $\omega_X = (\wedge^n \Omega_X^1)^{**}$

Ω_X^1 sheaf of Kähler differential

$n = \dim X$

ω_X is called the canonical sheaf of X.

Def: X is called $\mathbb{Q}$-factorial if

① all Weil divisor is $\mathbb{Q}$-Cartier

② $\exists m>0$ s.t. $(\omega_X^{\otimes m})^{**}$ is locally free.

Def: $\mathrm{Pic}(X) = \{$ locally free sheaf of rank one on X $\} / \cong$

$\mathrm{Pic}(X)$ is an abelian group.

We use additive notion for $\mathrm{Pic}(X)$.

Then we set $\mathrm{Pic}(X)_{\mathbb{Q}} = \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$

We note K_X the class of ω_X in $\mathrm{Pic}(X)_{\mathbb{Q}}$.

$$K_X = \frac{1}{m}(\omega_X^{\otimes m})^{**} \in \mathrm{Pic}(X)_{\mathbb{Q}}$$

Def: K_X is called a canonical divisor of X.

K_X might NOT be a Weil divisor in Kähler case.

Def: (~~terminal~~ klt singularities)

Let X be a normal $\mathbb{Q}$-factorial Kähler compact variety. Let $f: \hat{X} \to X$ be a resolution of sing

Then we can write $K_{\hat{X}} \sim_{\mathbb{Q}} f^* K_X + \sum a_i E_i$

in $\mathrm{Pic}(\hat{X})_{\mathbb{Q}}$

where $\{E_i\}$ are f-exceptional divisors.

We say that X has $\begin{cases} \text{terminal sing} & \text{if } a_i > 0 \\ \text{klt sing} & \text{if } a_i > -1. \end{cases}$

Differential form, Current on an analytic variety.

A (p,q)-form ω on X is a (p,q)-form on the smooth locus of X, such that for any $x \in X$, $\exists$ open neighbourhood $U \ni x$ s.t.

$x \in X$, $\exists$ open neighbourhood $U \ni x$ s.t.

$$U \overset{\varphi}{\simeq} V \underset{\substack{\text{analy}\\\text{subvar}}}{\subseteq} D \underset{\substack{\text{open}\\\text{domain}}}{\subseteq} \mathbb{C}^N$$

such that $\omega|_{U_{reg}} = \varphi^* \omega_D|_{V_{reg}}$ where ω_D is a (p,q)-form on D.

((p,q)-form is of the shape $\sum \varphi_{I,J}\, dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$)

Similarly, a (p,q)-current is a (p,q)-form with distribution coefficients.

Def: A form (or a current) ω is called closed, if $d\omega = 0$.

We can hence define pushforward of current and pullback of forms for a holomorphic morphism $f: X \to Y$.

Positive Current: $\dim X = n$ (positive means ≥ 0)

Def: A (p,p)-current Θ is positive if $\forall$ $(1,0)$-forms $\alpha_1, \cdots, \alpha_{n-p}$, the current

$$\Theta \wedge (\sqrt{-1}\,\alpha_1 \wedge \bar{\alpha}_1) \wedge \cdots \wedge (\sqrt{-1}\,\alpha_{n-p} \wedge \bar{\alpha}_{n-p})$$

is a positive measure.

Example: * Let $S \subseteq X$ be a closed analytic subset of X. of pure dimension $n-p$.

Then the integration on S is a positive (p,p)-current.

$$\alpha \longmapsto \int_S \alpha$$

* a closed real $(1,1)$-current Θ is positive $\Longleftrightarrow$

locally, $\Theta = \sqrt{-1}\,\partial\bar{\partial} u$, where u is a plurisubharmonic function.

Def: (Grauert). Let X be a (compact) complex analytic variety. Let ω be a $(1,1)$-form on X.

ω is a Kähler form if $\forall x \in X$, $\exists$ open neighbourhood $U \ni x$, s.t.

$$U \overset{\varphi}{\simeq} V \xhookrightarrow[\text{analy sub}]{\text{closed}} D \underset{\text{open domain}}{\subseteq} \mathbb{C}^N$$

s.t. $\exists$ Kähler form ω_D on D

s.t. $\omega|_U = \varphi^* \omega_D|_V$.

If this is the case, then (X, ω) is called a Kähler variety.

Note: ω is a Kähler form $\Leftrightarrow$ locally $\omega = \sqrt{-1}\partial\bar{\partial}u$ where u is a strictly pluri-subharmonic function.

<u>Remark</u>: * If (X, ω) is a Kähler variety, and if $f: \hat{X} \to X$ is a resolution of sing. Then $\exists$ Kähler form $\hat{\omega}$ on $\hat{X}$. This means $\hat{X}$ is also a Kähler variety. (Key: negativity lemma)

* A subvariety of a Kähler variety is a Kähler variety.

<u>Def</u>: (Fujiki class C) A complex ~~variety~~ variety X is called of class C if $\exists$ a Kähler variety Y such that $Y \to X$ is a bimeromorphic map.

$\boxed{\text{Bott-Chern Cohomology} \quad \text{Forms (Current) with local potential}}$ Boucksom-Eyssidieux-Guedj

This replaces the role of divisor in the Kähler case.