(III) $\boxed{\text{Curvature formula}}$ for direct images (Berndtsson 09, B-P-Wang).

- $\pi : X \longrightarrow Y$ $\qquad \pi_*(mK_{X/Y}+L) \geq 0$.
- If X_y has nontrivial deformatn $\rightsquigarrow$ we expect more positivity of $\pi_*(mK_{X/Y}+L)$.
- Kawamata 85 : $\pi : X \longrightarrow Y$ between two proj mfds.
  - $\mathrm{Var}(\pi) = \dim Y$, F generic fiber abundance
  - $\Rightarrow$ If $\pi_*(mK_{X/Y}) \neq 0 \Rightarrow \det \pi_*(mK_{X/Y})$ big on Y.
    line bundle
  - $\rightsquigarrow$ $\pi_*(mK_{X/Y})$ has many global sections.
- (Viehweg-Zuo, ...) $\pi : X_0 \longrightarrow Y_0$ be a proper ~~fibration~~ submersion between two quasi-proj mfds and $(*)$.
  - $\Rightarrow$ K_{Y_0} big ($Y_0 \subset Y$ $\Delta = Y \setminus Y_0$ snc.
    i.e., $K_Y + \Delta$ is big on Y.
  - Y_0 hyperbolicity .....

① Kodaira-Spencer map

π submersn, proper. X Kähler.

$$\boxed{\begin{array}{ccc} X_0 & \subset & X^{n+1} \\ \downarrow & & \downarrow \pi \\ 0 & \in & \Delta^1 \leftarrow \text{unit disc} \end{array}}$$

trivial line bundle on X_0

On X_0 : $0 \to T_{X_0} \to T_X|_{X_0} \xrightarrow{\pi_*} \pi^*(T_{\Delta^1, 0}) \to 0$

$$\begin{array}{ccc} X_0 & \subset & X^{n+1} \\ \downarrow & & \downarrow \pi \\ 0 & \in & (\Delta^1, t) \end{array}$$

$\frac{\partial}{\partial t}$

$\rightsquigarrow \rho : H^0(X_0, \pi^* T_{\Delta^1}) \to H^1(X_0, T_{X_0})$

$T_{\Delta^1, 0} = \mathbb{C} =$

We call ρ is the Kodaira-Spencer map.

$\boxed{\theta} := \rho(\frac{\partial}{\partial t}) \in H^1(X_0, T_{X_0})$.

Thm : $\theta(t) = 0 \in H^1(X_t, T_{X_t}) \quad \forall t \in \Delta^1$

$\Rightarrow \pi$ is locally trivial, i.e., $X_t \underset{\text{biholo}}{\sim} X_{t'} \quad \forall t, t' \in \Delta^1$.

- Calculate $\boxed{\theta}$ by differential forms

By partition of unity, $\exists\, V \in C^\infty(X, T_X)$

s.t. $\pi_*(V) = \frac{\partial}{\partial t}$

Then $\bar\partial V \in C^\infty(X, \Omega_X^{(0,1)} \otimes T_X)$.

Then $\bar\partial V|_{X_0} \in H^1(X_0, T_{X_0})$.

$\Rightarrow \theta = [\bar\partial V|_{X_0}] \in H^1(X_0, T_{X_0})$.

② $u \in H^0(\Delta, \pi_*(K_{X/\Delta} + L)) = H^0(X, K_{X/\Delta} + L)$

. C^∞ rep of u and its relation with θ.

$$\boxed{i\partial\bar\partial \|u\|_h^2}$$

$$\begin{array}{c} X^{n+1} \\ \downarrow \pi \\ t \in Y = \Delta^1 \end{array}$$

Let $u \in H^0(\Delta, \pi_*(K_{X/Y}+L)) = H^0(X, K_{X/Y}+L)$.

$\Rightarrow u \wedge \pi^* dt \in H^0(X, K_X + L)$.

Let $\boxed{V \in C^\infty(X, T_X) \text{ s.t. } \pi_*(V) = \frac{\partial}{\partial t}}$

Then $\tilde{u} := V \lrcorner \underbrace{(u \wedge \pi^* dt)}_{(n+1,0)-L} \in C^\infty_{(n,0)}(X, L)$.

and $\boxed{\tilde{u}|_{X_t} = u(t)} \in H^0(X_t, K_{X/Y}+L) = H^0(X_t, K_{X_t}+L)$

Def: Let $u \in H^0(\Delta, \pi_*(K_{X/Y}+L))$. Let $\tilde{u} \in C^\infty_{(n,0)}(X, L)$.

We say that $\tilde{u}$ is a C^∞-rep of u, if

$\tilde{u}|_{X_t} = u(t)$.

Exo 1. Let $\tilde{u}_1, \tilde{u}_2$ be two C^∞-rep of u

$\Rightarrow \boxed{\tilde{u}_1 - \tilde{u}_2 = \pi^*(dt) \wedge \phi}$, where $\phi \in C^\infty_{(n-1,0)}(X, L)$.

Exo 2: If $\tilde{u}$ is a $\boxed{C^\infty_{(n+1,p)}(X, L)} \Rightarrow \exists\, \phi \in C^\infty_{(n,p)}(X, L)$

s.t. $\tilde{u} = \pi^*(dt) \wedge \phi$.

Idea: $\bigcup_{i \in I} U_i = X$ $\quad \tilde{u} = \pi^* dt \wedge \phi_i$ on U_i

Let $\sum_i \theta_i = 1$. $\quad \tilde{u} = \pi^* dt \wedge \underbrace{(\sum \theta_i \phi_i)}$.

$$\begin{array}{c} X^{n+1} \\ \downarrow \pi \\ Y = \Delta^1 \ni t \end{array}$$

- relation with K-S.

Prop: Let $u \in H^0(\Delta, \pi_*(K_{X/Y} + L))$. Let $\hat{u} \in C^\infty_{(n,0)}(X, L)$ be a C^∞ rep of u.

Then $\exists\, \eta \in C^\infty_{(n-1,1)}(X, L)$, s.t. $\boxed{\bar\partial(\hat{u})} = (\pi^* dt) \wedge \eta$.

$\uparrow$ $(n,1)$-L

$\pi^* dt \wedge \eta_1 = \pi^* dt \wedge \eta_2$

$\qquad |_{X_t} \qquad\qquad |_{X_t}$

proof: $\bar\partial(\hat{u}) \wedge \pi^* dt$

$= \bar\partial(\hat{u} \wedge \pi^* dt)$

$\overset{\text{Exo}}{=} \bar\partial(\underbrace{u \wedge \pi^* dt}_{H^0(X,\, K_X + L)}) \equiv 0$.

$\Rightarrow \bar\partial(\hat{u}) = \boxed{(\pi^* dt) \wedge \eta}$. $\square$

Rk: η is not unique. But $\eta|_{X_t} \in C^\infty_{(n-1,1)}(X_t, L)$ is unique.

Prop: - $\eta|_{X_t}$ is $\bar\partial$-closed form. $\Rightarrow [\eta|_{X_t}] \in \boxed{H^1(X_t, \Omega^{n-1}_{X_t} \otimes L)}$

- $[\eta|_{X_t}]$ is indepent of choice of C^∞ rep $\boxed{\hat{u}}$

proof: $\bar\partial \circ \bar\partial = 0$. $\Rightarrow \underbrace{\bar\partial(\bar\partial \hat{u})}_{0=} = \bar\partial(\pi^* dt \wedge \eta)$

$= \pi^* dt \wedge (\bar\partial \eta)$

$\Rightarrow \bar\partial \eta|_{X_t} = 0$.

- Let $\hat{u}_1, \hat{u}_2$ be two C^∞ rep of u.

$\Rightarrow \hat{u}_1 = \hat{u}_2 + \pi^*(dt) \wedge \phi \quad \phi \quad (n-1,0)-L.$

$\Rightarrow \bar{\partial}(\hat{u}_1) = \bar{\partial}(\hat{u}_2) + \pi^* dt \wedge \bar{\partial}\phi$

$\qquad \rotatebox{90}{$=$} \qquad\qquad \rotatebox{90}{$=$}$

$\pi^*(dt) \wedge \eta_1 \quad \pi^*(dt) \wedge \eta_2$

$\bar{\partial}(\hat{u}) = dt \wedge \eta$

$\Rightarrow (\eta_1 - \eta_2)|_{X_t} = [\bar{\partial}\phi]|_{X_t} = \bar{\partial}(\phi|_{X_t}).$

Prop: Let ω_X be a Kähler class on X^{n+1}.

Then:

- $\boxed{[\eta|_{X_t}]} = [\theta] \wedge [u(t)] \in H^1(X_t, \Omega^{n-1}_{X_t} \otimes L)$

$\qquad [\theta] \in H^1(X_t, T_{X_t}), \quad [u(t)] \in H^{0}(X_t, K_{X_t}+L)$

- $\boxed{[\eta|_{X_t}] \wedge [\omega_X|_{X_t}] = 0}$

- $\exists\ \hat{u}$ C^∞ rep of u, s.t. $\bar{\partial}\hat{u} = dt \wedge \eta$

and $\eta|_{X_t}$ is $\omega_X|_{X_t}$-primitive. i.e. $(\eta \wedge \omega_X)|_{X_t} = 0$

proof: $\exists\ V \in C^\infty(X, T_X)$ s.t. $\pi_*(V) = \frac{\partial}{\partial t}$

$\hat{u} := V \lrcorner (\underbrace{u \wedge \pi^* dt}_{\text{holo}})$ is a C^∞ rep.

$\bar{\partial}(\hat{u}) = (\bar{\partial}V) \lrcorner (u \wedge \pi^* dt) \Rightarrow \eta|_{X_t} = [\bar{\partial}V]|_{X_t} \wedge u.$

θ

$(n,0)$ $\qquad$ $\mathrm{Dirt}(\alpha)$

Let $\tilde{u}$ be a C^∞ rep of u.

$$\Rightarrow \partial(\underbrace{\tilde{u}\wedge w_X}_{}) = \partial(\tilde{u})\wedge w_X = \underline{dt\wedge\eta\wedge w_X}$$

$\uparrow$ Kähler

$[\eta]=0$

$[\eta]=[\theta]\wedge[u]$

$(n+1,1)-L$

$\|$

$\iota^*(dt)\wedge\phi \qquad \Rightarrow (\bar\partial\phi)|_{X_t} = (\eta\wedge w_X)|_{X_t}$

$\|$ $\qquad\qquad\qquad\qquad \|$

$\underline{\iota^*(dt)\wedge(\partial\phi)} \qquad \underline{\partial(\phi|_{X_t})}$

③ Curvature of direct images:

Recall: (E, h_E) is a holo vector bundle on X

$\uparrow$ herm metric.

$$u\in C^\infty_{(p,q)}(X,E),\quad v\in C^\infty_{(p',q')}(X,E)$$

$$\{u,v\}\in C^\infty_{p+q',q+p'} \qquad u=\sum u_i\otimes e_i,\quad u_i\in(p,q),\ e_i\in E$$

$$\| \qquad\qquad v=\sum v_i\otimes e_i,\quad v_i\in(p',q'),\ e_i\in E$$

$$\sum_{i,j} u_i\wedge\overline{v_j}\,\langle e_i,e_j\rangle_{h_E}$$

- $\dim X=n$: $\quad$ - $\underline{u\in C^\infty_{(n,0)}(X,E)}\Rightarrow \underline{c_n\{u,u\}\geq 0}$ $\quad(n,n)$

$\exists\, c_n = i^{n^2}$

- $\underline{u\in C^\infty_{(n-1,1)}(X,E)+ u\wedge w_X\equiv 0} \Rightarrow \exists\, c_n'$, $\underline{c_n'\cdot\{u,u\}\geq 0}$.

$i\Theta_{h_L}(L) \geq 0$ $\quad X^{n+1}$

$\downarrow$

$\cdot\ u \in H^0(\Delta^1, \pi_*(K_{X/Y}+L))$, $\quad t \in \Delta^2$ $\quad t \in \Delta^2$

$\|u\|_h^2(t) := c_n \int_{X_t} \{u, u\}$ $\quad u|_{X_t} \in H^0(X_t, K_{X_t}+L)$,

Thm (Berndtsson): $X^{n+1} \xrightarrow{\text{proper submersion}} \Delta^1$ $\quad X^{n+1}$ Kähler

$L \to X$ $\quad h_L\ C^\infty$

Let $u \in H^0(\Delta^2, \pi_*(K_{X/Y}+L))$.

Then $\exists\ \tilde{u}^{(n,0)}$ C^∞ rep of u, s.t. $\bar\partial \tilde{u} = dt \wedge \eta$

$\eta|_{X_t}$ is $\omega_X|_{X_t}$-primitive and

$$\underbrace{\boxed{\langle i\Theta_h(u), u\rangle_h(0)}}_{(1,1)\text{-form on } \Delta} = \underbrace{c_n\, \pi_*\{i\Theta_{h_L}(L)\wedge\tilde{u}, \tilde{u}\}}_{(n+1,n+1)\text{-form on } X} + c_n'\, \pi_*\{\eta, \eta\}\ i\,dt\wedge d\bar{t}$$

(I) $\leftarrow$ (1,1)-form

(II)

$(n-1,1)$-L $\quad (n,n)$

RK.

- If $i\Theta_{h_L}(L) \geq 0 \Rightarrow (\mathrm{I}) \geq 0$.
- $\eta|_{X_t}$ $\omega_X|_{X_t}$ primitive $\Rightarrow (\mathrm{II}) \geq 0$

$\Rightarrow$ If $i\Theta_{h_L}(L) \geq 0 \Rightarrow \langle i\Theta_h(u), u\rangle_h(0) \geq 0$.

$X \xrightarrow{\pi} Y$ is proper fibration between two Kähler mfld.

thm1

- $(\pi_*(K_{X/Y}+L), h) \succeq 0$ on Y.
- If ~~$\forall$~~ $\exists\, y \in Y$, ~~s.t.~~ π is a submersion near y, and $i\Theta_h(\pi_*(K_{X/Y}+L)) > 0$ on y.

$\Rightarrow$ $\pi_*(K_{X/Y}+L)$ big $\Rightarrow$ $\mathrm{Sym}^m \pi_*(K_{X/Y}+L)$ has a lot of sections.

- When $i\Theta_h(\pi_*(K_{X/Y}+L)) > 0$ on some pt $y \in Y$?
  - If $i\Theta_{h_L}(L)$ is strictly positive in some horizontal direction.
    $\Rightarrow i\Theta_h(\pi_*(K_{X/Y}+L)) > 0$
  - If $[\eta|_{X_y}] \neq 0$ $\Rightarrow$ $i\Theta_h(\pi_*(K_{X/Y}+L))^{(n,n)} > 0$.
    $\| [\theta]\wedge[u(y)]$

- $X \to Y$, F | K_F semi-ample.

$\pi_*(mK_{X/Y})$

$\mathbb{P}(\pi_*(mK_{X/Y})) \to Y$, $X \to Y$

proof: Step 1: Let $\hat{u}$ be a C^∞ rep of $u \in H^0(\Delta, \pi_*(K_{X/\Delta} \otimes L))$

s.t $\bar\partial \hat{u} = i(dt) \wedge \eta$

and η $(n-1,1)$-form $-L$ $\quad \eta|_{X_t}$ ω_X-primitive.

$$\|u\|_h^2(t) = c_n \int_{X_t} \{u, u\}$$

$$\Rightarrow \|u\|_h^2 = c_n \pi_* \underbrace{\{\hat{u}, \hat{u}\}}_{(n,n)} \quad \text{as a function on } \Delta^1.$$

$X^{n+1} \to \Delta^1$

$\langle i\Theta_h(u), u\rangle_h = ?$

$$i\partial\bar\partial \|u\|_h^2 = \boxed{\langle D_h' u, D_h' u\rangle} - \langle i\Theta_h(u), u\rangle_h$$

$$\Rightarrow \langle i\Theta_h(u), u\rangle_h = \langle D_h' u, D_h' u\rangle - i\partial\bar\partial \|u\|_h^2$$

$$= \langle D_h' u, D_h' u\rangle - i\partial\bar\partial\, c_n \pi_* \{\hat{u}, \hat{u}\}$$

$$= \langle D_h' u, D_h' u\rangle - c_n \pi_* i\partial\bar\partial \{\hat{u}, \hat{u}\}$$

$$\partial\{\hat{u}, \hat{v}\} = \{D_h' \hat{u}, \hat{v}\} + (-1)^{\deg \hat{u}} \{\hat{u}, \bar\partial \hat{v}\}$$

Step 2:
$$\begin{cases} \bar\partial \hat{u} = dt \wedge \eta & \eta \ (n-1,1)\text{-}L \\ \underbrace{D_{h_L}' \hat{u}}_{(n+1,0)} = dt \wedge \boxed{\mu} & \mu \ (n,0)\text{-}L \end{cases}$$

$$\langle i\Theta_h(u), u\rangle_h = C_n \pi_* \{ i\Theta_{h_2}(L)(\tilde{u}), \tilde{u} \}$$

$$+ C_n' \pi_* \{\eta, \eta\}\, i\, dt \wedge d\bar{t}$$

$$+ \boxed{C_{n+1} \{ D'_{h_2}\tilde{u}, D'_{h_2}\tilde{u} \} - C_{n+1}\pi_* \{\mu, \mu\}\, dt \wedge d\bar{t}}$$

<u>Step 3</u>: $\exists\ \tilde{u}$ C^∞ rep. s.t. $\eta|_{X_t}$ ω_X-primitive

$\mathrm{(II)} \leq 0$

$$\boxed{D'_{h_2}\tilde{u}|_{X_t} = \mu \cdot dt}$$

$$\mu|_{X_t} \in C^\infty_{(n,0)}(X_t, L) = \underbrace{\ker \bar\partial}_{= H^0(X_t, K_{X_t}+L)} \oplus \operatorname{Im} \bar\partial^*$$

$$\mu|_{X_t} = \underbrace{P_1(\mu|_{X_t})}_{\in \ker\bar\partial} + \underbrace{P_2(\mu|_{X_t})}_{\in \operatorname{Im}\bar\partial^*} \in \operatorname{Im}\bar\partial^*$$

$$= \bar\partial^* a = [D', \Lambda_\omega]a = D'(\Lambda_\omega a)$$

- $\underline{D'_h u(t) = P_1(\mu|_{X_t})\, dt}$

- $\mathrm{(II)} = -\|P_2(\mu)\|^2_{L^2}\, dt \wedge d\bar{t} \leq 0$, $\quad P_2(\mu) = 0$

$$\tilde{u} \rightsquigarrow \underline{\tilde{u} - dt \wedge (\Lambda_\omega a)}$$

$\square$

· Higher direct images (BPW): $\dim X_y = n$

$\pi: X \longrightarrow Y$ be a proper submersion. X, Y Kähler.

Suppose: · $L \to X$ holo. $\left(i\Theta_{h_L}(L)|_{X_y} \begin{matrix} >0 \\ <0 \end{matrix} \quad h_L\ C^\infty \right)$.

· $\underline{R^i\pi_*(\Omega^{n-i}_{X/Y} \otimes L)}$ is locally free

and $\left[R^i\pi_*(\Omega^{n-i}_{X/Y}\otimes L)\right]_y = H^{n-i,i}(X_y, L)$.

$\underline{\text{Thm}}$: $\exists$ $\underline{h}$ herm C^∞ metric on $R^i\pi_*(\Omega^{n-i}_{X/Y}\otimes L)$

and $\underline{i\Theta_h(R^i\pi_*(\Omega^{n-i}_{X/Y}\otimes L))} = \underline{\cdots}$

$\boxed{\text{Thm}}$: If $i\Theta_{h_L}(L) \le 0$.

· $\tau: \boxed{R^i\pi_*(\Omega^{n-i}_{X/Y}\otimes L)} \xrightarrow{\lambda\rho} R^{i+1}\pi_*(\Omega^{n-i-1}_{X/Y}\otimes L)\otimes \Omega^1_Y$

· Let $\boxed{N_i := \ker\tau}$.

$\underline{\left(\left(N_i/\text{torsion}\right)_{\text{locally free}}, h \right) \le 0}$ on Y_0 locally free locus. $\square$