Bott-Chern Cohomology

$\partial f = \sum_{i=1}^{n} \frac{\partial f}{\partial z_i} dz_i$, $\bar\partial f = \sum_{i=1}^{n} \frac{\partial f}{\partial \bar z_i} d\bar z_i$

Let X be a normal compact Kähler variety, with rational singularities

Def ($\mathcal{H}_X$): Let $\mathrm{Re}\,\mathcal{O}_X$ be the sheaf of real parts of holomorphic functions.

Let $\mathcal{H}_X = \sqrt{-1} \cdot \mathrm{Re}\,\mathcal{O}_X$

Lemma: Assume that θ is a $(0,0)$-current on X. If $\partial\bar\partial\theta = 0$, then θ is a continuous function.

Indeed, in this case, θ is a pluriharmonic function.

(Lemma 4.6.1 in the book of Boucksom-Eyssidieux-Guedj)

A pluriharmonic function is always the real part of a holomorphic function.

Def: A real closed $(1,1)$-current θ is called with local potential if locally, there exists current u s.t. $\theta = \sqrt{-1}\,\partial\bar\partial u$

Remark: While patching local expressions $\sqrt{-1}\,\partial\bar\partial u$, $\sqrt{-1}\,\partial\bar\partial v$ we need $\sqrt{-1}\,\partial\bar\partial(u-v) = 0$.

This means that $u-v$ is pluriharmonic

<u>Remark</u>: To compare with the definition of Cartier divisor, in Hartshorne. (A Cartier divisor is a global section of $K(X)/\mathcal{O}_X^*$)

<u>Def</u>: A real closed (1,1)-current with local potential can be also defined as a global section of the sheaf

$$\mathcal{D}_X / \mathcal{H}_X .$$

Here $\mathcal{D}_X$ is the sheaf of real distribution.

First, we have the exact sequence.

$$0 \to \mathcal{H}_X \to \mathcal{D}_X \to \mathcal{D}_X/\mathcal{H}_X \to 0$$

$\Rightarrow$ long exact sequence

$$\cdots H^0(X,\mathcal{D}_X) \to H^0(X,\mathcal{D}_X/\mathcal{H}_X) \to H^1(X,\mathcal{H}_X) \to H^1(X,\mathcal{D}_X) \cdots$$

Since $\mathcal{D}_X$ is acyclic, thus $H^1(X,\mathcal{D}_X)=0$.

Hence we have a surjection

$$H^0(X, \mathcal{D}_X/\mathcal{H}_X) \twoheadrightarrow H^1(X,\mathcal{H}_X)$$

whose kernel are (1,1)-current which is globally $\partial\bar\partial$-exact

<u>Def</u>: $N^1(X) = H^1(X,\mathcal{H}_X) := H^1_{BC}(X)$

Remark: $N^1(X)$ is the Kähler analogue of the space of divisors in the algebraic case.

We have the following exact sequence.

$$0 \to \sqrt{-1}\mathbb{R} \to \mathcal{O}_X \to \mathrm{Re}\,\mathcal{O}_X \to 0$$

We multiply by $\sqrt{-1}$, then we get

$$0 \to \mathbb{R} \to \mathcal{O}_X \to \mathcal{H}_X \to 0$$

We deduce a long exact sequence

$$\cdots H^1(X,\mathbb{R}) \to H^1(X, \mathcal{O}_X) \to \underset{\substack{\| \\ N^1(X)}}{H^1(X,\mathcal{H}_X)} \to H^2(X,\mathbb{R})$$

When X is smooth compact Kähler, we can use Hodge theory to show that $H^1(X,\mathbb{R}) \to H^1(X,\mathcal{O}_X)$ is an iso.

Then $H^1(X,\mathcal{H}_X) \to H^2(X,\mathbb{R})$ is an injection.

Indeed, $H^1(X,\mathcal{H}_X) \simeq H^{1,1}(X)$

When X has rational singularities, we can also show that

$H^1(X,\mathcal{H}_X) \hookrightarrow H^2(X,\mathbb{R})$ is an injection.

Remark: In general, when X is a smooth proj variety, the space $N^1(X)$ is strictly larger than

$$NS(X)_{\mathbb{Q}} = \{\mathbb{Q}\text{-divisors}\}/\equiv$$

For example, rank $NS(X)_{\mathbb{Q}} = \rho(X)$ is the Picard number

$\dim_{\mathbb{Q}} N^1(X) = h^{1,1}(X)$.

For K3 surfaces, there are K3 surfaces with $\rho(X) < h^{1,1}(X)$

Positivity of (1,1) current, of Bott-Chern class.

<u>Def</u>: A class $\alpha \in N^1(X)$ is called pseudo-effective (or psef) if α can be represented by a current T, such that locally $T = \sqrt{-1}\partial\bar{\partial} u$ where u is a psh function.

<u>Def</u>: A class $\alpha \in N^1(X)$ is called nef, if it can be represented by a FORM T such that for any Kähler form ω, $\exists$ a C^∞ function f_ω such that $T + \sqrt{-1}\partial\bar{\partial} f_\omega + \omega \geq 0$ is pseudo-effective.

<u>Def</u>: We set $\overline{Nef(X)} \subseteq N^1(X)$ the closed convex cone generated by nef classes.

Thm: When X is normal Kähler, then $\overline{Nef(X)}$ is the closure of $K(X)$, where $K(X)$ is the collection of Kähler classes.

Recall that for a smooth projective variety X, X is called minimal if K_X is nef.

Dual Kähler (nef) cone $\overline{NA}(X)$

We consider the space

$N_1(X) = \{ \text{real closed } (n-1, n-1)\text{-current} \} / \equiv$

where $T_1 \equiv T_2 \iff T_1(\eta) = T_2(\eta)$ for all $\eta \in N^1(X)$.

Then $N^1(X)$ is dual to the space $N_1(X)$.

We set $\overline{NA}(X) \subseteq N_1(X)$ the closed convex cone generated by positive currents.

Recall that $\overline{NE}(X) \subseteq N_1(X)$ is the cone generated by curve classes.

($\overline{NE}(X)$ is called the Mori cone).

Naturally, $\overline{NE}(X) \subseteq \overline{NA}(X)$.

Theorem (Demailly-Păun) : X compact normal of class $\mathcal{C}$. Then $\overline{NA}(X)$ is dual to $\overline{Nef}(X)$

<u>Remark</u>: In Kähler case, it is not clear how the dual cone of $\overline{NE}(X)$ looks like

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Cone Theorem for $\overline{NA}(X)$: K_X pseudo-eff.

(†) X compact Kähler $\mathbb{Q}$-fac, 3-fold, terminal sing.

Then $\exists\, d>0$, $\exists$ at most countablely many curves Γ_i with $0 < -K_X \cdot \Gamma_i \le d$ s.t.

$$\overline{NA}(X) = \overline{NA}(X)_{K_X \ge 0} + \sum_i \mathbb{R}^+ [\Gamma_i]$$

Divisorial Zariski decomposition (Boucksom)

Def: X a compact Kähler manifold. α a psef class in $N^1(X)$.

Then $\exists!$ decomposition

$$\alpha = N(\alpha) + \sum_{i=1}^{r} \nu_i D_i$$

where $N(\alpha)$ is nef in codimension one

and $\nu_i > 0$, D_i is a prime divisor

Furthermore, $r \le \rho(X)$.

Nef in codim one: For any prime divisor $S \subseteq X$ $N(\alpha)|_S$ is pseudo-effective.

Adjunction Assume X satisfies $(+)$

Let $S \subseteq X$ be prime divisor.

Then S is a surface.

Let $\pi: \hat{S} \longrightarrow S$ be the minimal resolution of singularities.

Then $K_{\hat{S}} \sim_{\mathbb{Q}} \pi^* K_S - E$, where E is an effective divisor in $\hat{S}$.

"Roughly, it mean $K_{\hat{S}}$ is more negative than K_S"

For example. Let $C \subseteq S$ be curve, which is not contained in the singular locus of S. Set $\hat{C}$ the strict transform of C in $\hat{S}$.

Then $K_{\hat{S}} \cdot \hat{C} \leq K_S \cdot C$.

Nefness criterion

In algebraic case, a divisor D is nef if $\forall$ subvariety $Y \subseteq X$, $D^{\dim Y} \cdot Y \geq 0$.

We can prove that D is nef $\Leftrightarrow$ $D \cdot C \geq 0$ $\forall$ curve C.

Demailly-Păun: X compact Kähler mfd, α a classe in $N^1(X)$. α is nef if and only if, for any subvariety $Y \subseteq X$,

$$\alpha|_Y \text{ is pseudo-effective.}$$

Notation: Assume X satisfies $(+)$

K_X psef $\Rightarrow$ Zariski decomposition

$$K_X = \sum_{i=1}^{r} \lambda_i S_i + \underset{\substack{\uparrow \\ \text{nef in codim one}}}{N(K_X)}$$

Lemma: Let $S \subseteq X$ be a prime divisor. Assume that $K_X|_S$ is not pseudo-eff.

Then ① S is one of the S_j.

② S is Moishezon and the minimal resolution $\hat{S}$ of S is a uniruled surface.

Proof: ① Assume that $S \notin \{S_1, \cdots, S_r\}$.

Then $$K_X|_S = \sum_{j=1}^{r} \lambda_j \cdot \underset{\substack{\uparrow \\ \text{effective}}}{(S_j \cap S)} + \underset{\substack{\uparrow \\ \text{psef}}}{N(K_X)|_S}$$

is psef. This is a contradiction.

② We may assume that $S = S_1$

Then $$S = S_1 = \frac{1}{\lambda_1} K_X - \frac{1}{\lambda_1}\left(\sum_{j>1} \lambda_j S_j + N(K_X)\right)$$

$$K_S = (K_X + S)|_S = \left(1 + \frac{1}{\lambda_1}\right) K_X|_S - \frac{1}{\lambda_1}\left(\sum_{j>1} \lambda_j (S_j \cap S)\right)$$

$$- \frac{1}{\lambda_1} (N(K_X))$$

$$-\frac{1}{\lambda_1}(N(K_X))|_S$$

$K_X|_S$ is not psef, $S_j \cap S$ is eff for $j > 1$

$N(K_X)|_S$ is psef

Thus K_S is not psef. By the section of Adjunction above, we see that $K_{\hat{S}}$ is not pseudo-eff.

Thus, since $\hat{S}$ is smooth, $\hat{S}$ is uniruled.

Thus $\hat{S}$ is algebraic. Hence S is Moishezon

Cor : X satisfies $(+)$. Then

K_X is nef $\Longleftrightarrow$ $K_X \cdot C \geq 0$ for all curve $C \subseteq X$.

Proof : Exercise. We only need to refine the proof of ② of the previous Lemma.

Proof of the Cone Theorem.

We will first show that

$$\overline{NE}(X) = \overline{NE}(X) \quad + \quad \sum \mathbb{R}^+[R_i]$$

$K_{X}\geq 0$

$\overline{NE}(X)$ is the CLOSED cone generated by curve classes.

Thus by definition,

$$\overline{NE}(X) = \overline{NE(X)_{K_X\geq 0} + \sum \mathbb{R}^+[\Gamma_i]} \leftarrow \text{closure}$$

<u>Lemma</u>: Let $N \subseteq \overline{NA}(X) \subseteq N_1(X)$ be a closed convexe cone. Assume that there are at most countable set I and curves $\{\Gamma_i\}_{i\in I}$, $\exists d>0$ s.t.

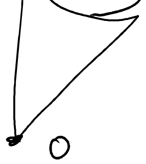
$0 < -K_X \cdot \Gamma_i \leq d$, and

$$N = \overline{N_{K_X\geq 0} + \sum \mathbb{R}^+[\Gamma_i]}$$

Then $N = N_{K_X\geq 0} + \sum \mathbb{R}^+[\Gamma_i]$.

<u>Proof</u>: set $V = N_{K_X\geq 0} + \sum \mathbb{R}^+[\Gamma_i]$. We need to show that V is closed.

Since $V \subseteq NA(X)$, V does not contain any line



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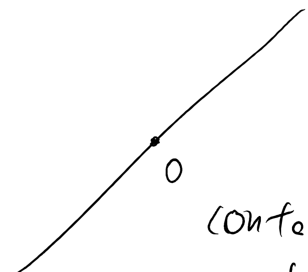
contains no line



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contains a line

Thus let $\mathbb{R}^+[r]$ be an extremal ray of $\overline{V}$.
It is enough to show that $[r] \in V$.
If $K_X \cdot r \geq 0$, then $r \in N_{K_X \geq 0} \subseteq V$. We may then assume $K_X \cdot r < 0$.
Let ω be a Kähler class. Let $\varepsilon > 0$ s.t.

$$(K_X + \varepsilon\omega) \cdot r < 0$$

By definition

$$N = \overline{N_{(K_X+\varepsilon\omega) \geq 0} + \sum_{j \in J} \mathbb{R}^+[\Gamma_j]}$$

where Γ_j are those curves in $\{\Gamma_i\}_{i \in I}$ with

$$(K_X + \varepsilon\omega) \cdot \Gamma_j < 0.$$

$$\text{“}(K_X+\varepsilon\omega) \cdot \Gamma_j < 0\text{”} + \text{“}0 < -K_X \cdot \Gamma_j \leq d\text{”}$$

$$\Rightarrow \quad 0 < \omega \cdot \Gamma_j \leq \frac{d}{\varepsilon}$$

$\Rightarrow$ J is a finite set.

$$J = \{1, \cdots, q\}.$$

Now $r \in N \Rightarrow r = \lim_{m \to \infty} (t_m + s_m)$

where $t_m \in N_{(K_X+\varepsilon\omega) \geq 0}$, $s_m \in \sum_{j \in J} \mathbb{R}^+[\Gamma_j]$

One can then show that, up to a subsequence,
$\lim t_m = 0$, and $r = \lim s_m$.

J is a finite set $\Rightarrow$ $[\Gamma] \in \mathbb{R}^+[\Gamma_j]$ for some $j \in J$.

This completes the proof.

We now assume the following Bend-and-Break Thm, and prove the cone theorem first.

Thm (B-and-B). Let X as $(\dagger)$.
Then $\exists\, d > 0$ s.t. If C is a curve s.t. $-K_X \cdot C > d$ then $[C] = [C_1] + [C_2]$, where C_1, C_2 are effective 1-cycles with $\mathbb{Z}$ coefficients.

Prop: X as $(\dagger)$ $\exists d > 0$. $\exists$ at most countably many curves Γ_i s.t. $0 < -K_X \cdot \Gamma_i \le d$ s.t.

$$\overline{NE}(X) = \overline{NE}(X)_{K_X \ge 0} + \sum \mathbb{R}^+[\Gamma_i].$$

Proof: We know that $\overline{NE}(X) = \overline{\overline{NE}(X)_{K_X \ge 0} + \sum \mathbb{R}^+[\Theta_\lambda]}$

where Θ_λ are curves with $-K_X \cdot \Theta_\lambda > 0$.

Bend-and-Break $\Rightarrow$

$$\overline{NE}(X) = \overline{\overline{NE}(X)_{K_X \ge 0} + \sum_{i \in I} \mathbb{R}^+[\Gamma_i]}$$

with $0 < \Gamma_i \cdot (-K_X) \leq d$.

I is at most countable by the proof of previous lemma.

By previous lemma, we deduce that

$$\overline{NE}(X) = \overline{NE}(X)_{K_X \geq 0} + \sum \mathbb{R}^+[\Gamma_i].$$

Proof of cone theorem for $\overline{NA}(X)$.

Set $V = \overline{NA(X)}_{K_X \geq 0} + \sum_{i \in I} \mathbb{R}^+[\Gamma_i]$

where Γ_i are all curves s.t. $0 < -K_X \cdot \Gamma_i \leq d$.

We need to show that $\overline{NA(X)} = V$.

The lemma above shows that, it is enough to show that $\overline{NA(X)} = \overline{V}$.

Since evidently $\overline{V} \subseteq \overline{NA}(X)$, we will show that $\overline{NA}(X) \subseteq \overline{V}$.

A theorem of Demailly-Păun shows that $\overline{NA}(X)$ is generated by 3 kinds of classes: $[\omega]^2$, $[C]$, $[\omega][S]$

where ω is any Kähler form, C is any curve and S is any surface.

1^{st} case, $\alpha = [\omega]^2$.

K_X is psef $\Rightarrow$ $K_X \cdot \alpha \geq 0 \Rightarrow \alpha \in \overline{NA}(X)_{K_X \geq 0} \subseteq \overline{V}$

OK

2^{nd} case $\alpha = [C]$. By previous proposition,

$$\alpha = [C] \in \overline{NE}(X)_{K_X \geq 0} + \Sigma \mathbb{R}^+[\Gamma_i] \subseteq \overline{V}. \quad \text{OK.}$$

3^{rd} case $\alpha = [\omega] \cdot [S]$.

If $K_X \cdot \alpha \geq 0$, then it is OK.

Assume that $K_X \cdot \alpha < 0$. We will show that

$\alpha \in \overline{NE}(X) \subseteq \overline{V}$

↑ by previous prop.

$K_X \cdot \alpha < 0 \Rightarrow (K_X|_S) \cdot (\omega|_S) < 0$

ω Kähler $\Rightarrow$ $K_X|_S$ is not psef.

Then S is one of the S_j of the Zariski dec.

Let $\pi_j : \hat{S}_j \to S_j$ be the minimal resolution.

Then $\hat{S}_j$ is uniruled, hence projective.

Furthermore, $H^2(\hat{S}_j, \mathcal{O}_{\hat{S}_j}) = 0$.

This implies that $[\pi_j^* \omega]$ is represented by a real divisor, which is big and nef.

$(H^1(\hat{S}_j, \mathcal{O}_{\hat{S}_j}^*) \to H^2(\hat{S}_j, \mathbb{Z}) \to H^2(\hat{S}_j, \mathcal{O}_{\hat{S}_j}) = 0$ is exact)

Thus $[\pi_j^* \omega]$ is a limit of curves $\{H_m\}$ on $\hat{S}_j$

$$[\pi_j^* \omega] = \lim H_m \quad \text{in } N_1(\hat{S}_j)$$

Then we can show that

$$\alpha = [\omega] + [S] = \lim [\pi(H_m)] \in \overline{NE}(X).$$