$$\Rightarrow \begin{cases} \circ\ b) \Rightarrow L\cdot E - E^2 \le 1 & ① \\ \circ\ L\cdot E \geq 0 & ② \\ \circ\ L^2 \cdot E^2 \le (L\cdot E)^2 & ③ \\ \circ\ J \Rightarrow L^2 \geq 2(L\cdot E). & ④ \\ \circ\ L^2 \geq 5 & ⑤ \end{cases}$$

$$L\cdot E \le 1 + E^2 \le 1 + \frac{(L\cdot E)^2}{L^2} \le 1 + \frac{(L\cdot E)}{2} \Rightarrow \underline{L\cdot E \le 2}$$

$\circ\ = 2 \Rightarrow L^2 \cdot E^2 = (L\cdot E)^2 = 4$ ξ.

$\Rightarrow (L\cdot E) \le 1$. if $L\cdot E = 1. \Rightarrow E^2 = 0$ $\underline{①+③}$

$L\cdot E = 0 \Rightarrow E^2 = \begin{cases} \cancel{0} \\ -1 \end{cases}$ $\underline{\text{HIT}}$

Hyperkähler manifolds.

X: compact Kähler manifold $c_1 = 0$

$\uparrow \pi$: étale (Beauville-Bogomolov decomposition theorem)

$$\tilde{X} = \underset{\substack{\uparrow \\ \text{complex torus} \\ \text{(Abelian var)}}}{A} \times \prod_i \underset{\substack{\uparrow \\ \text{Calabi-Yau}}}{X_i} \times \prod_j \underset{\substack{\uparrow \\ \text{hyperkähler (HK)}}}{Y_j}$$

X: CY $\overset{}{\underset{\text{def}}{\Longleftrightarrow}} \begin{cases} X: \text{simply connected},\ K_X \sim 0 \\ h^i(X, \mathcal{O}_X) = 0 \quad 0 < i < \dim X \end{cases}$

$\underset{\dim = 2n}{X: \text{HK}} \overset{\text{def}}{\Longleftrightarrow} \begin{cases} X: \text{simply conn.}\ (K_X \sim 0) \\ H^0(X, \Omega_X^2) = \mathbb{C}\langle\sigma\rangle \quad \sigma: \text{every where non-degenerate 2-form.} \end{cases}$

$(\rightsquigarrow H^0(X, \omega_X) = \mathbb{C}\langle \underline{\underline{\sigma^n}} \rangle \Rightarrow \omega_X \cong \mathcal{O}_X)$.

Example: $\cdot$ K3 surfaces

- $K3^{[n]}$: Hilbert sch of n pts on K3
- $K_n(A)$: generalized Kummer
- OG6, OG10 by O'Grady

$K3^{[n]}$: S : K3 surface. $n \in \mathbb{Z}_{>0}$

$$S^{[n]} := \text{moduli space of } \boxed{n \text{ pts}} \text{ on } S$$

$$\overset{\shortparallel}{} \quad Z \subseteq S \quad \text{0-dim subsch. with length}(\mathcal{O}_Z) = n.$$

$n=2$

$$\begin{tikzcd}
\{(x,y)\} \in S \times S \arrow[d, "/\mathbb{Z}_2"] & \widetilde{S \times S} \arrow[l, "\text{blow-}\Delta"'] \arrow[d, "\mathbb{Z}_2"] \\
\{\{x,y\}\} = \mathrm{Sym}^2(S) & \boxed{S^{[2]}} \arrow[l]
\end{tikzcd}$$

Cannot distinguish $Z \subseteq S$ with $\mathrm{Supp} Z = x$

S : K3 $\Longrightarrow$ $S^{[n]}$: HK of dim $2n$

S : Abelian $\Longrightarrow$ $S^{[n]} \xrightarrow{x_1 + \cdots + x_n} S$

$\cup$

$\mathrm{Ker} := K_{n-1}(S)$: HK of dim $2n-2$.

BBF form = Beauville-Bogomolov-Fujiki form)

$\exists\, q_X : H^2(X, \mathbb{R}) \to \mathbb{R}$ quadratic form

$\exists\, c_X \in \mathbb{R}_{>0}$ s.t.

$$\forall \alpha \in H^2(X, \mathbb{R}), \quad \alpha^{2n} = c_X (q_X(\alpha))^n$$

In particular, $L^{2n} = c_X \cdot (q_X(L))^n$. $\forall L$: divisor on X.

q_X, c_X unique if further

$\begin{cases} \circ\ q_X : \text{primitive integral on } H^2(X,\mathbb{Z}) \\ \circ\ q_X(\sigma+\bar\sigma)>0 \end{cases}$

$q_X \rightsquigarrow q_X(-,-) : H^2(X,\mathbb{R}) \times H^2(X,\mathbb{R}) \longrightarrow \mathbb{R}$

bilinear form (~ "intersection" of divisors on HK)

- $q_X(-,-)$ has sign $(1, b_2-3)$ on $H^{1,1}(X)$

($\Rightarrow$ if H, E : divisors ($E \neq 0$)

HIT. $q_X(H)>0,\ q_X(H,E)=0 \Rightarrow q_X(E)<0$)

Fact: $\begin{cases} \circ\ L:\text{ample} \Rightarrow q_X(L)>0 \\ \circ\ L:\text{big} \not\Rightarrow \quad (\text{big} \Leftrightarrow h^0(X,mL) \sim cm^{2n}) \\ \circ\ E,F:\text{effective div without common components} \end{cases}$

$\Rightarrow q_X(E,F) \geq 0$ (Exercise)

Ex $\int_X \alpha_1 \cdots \alpha_{2n} = \frac{c_X}{(2n)!} \sum_{\sigma \in S_{2n}} q_X(\alpha_{\sigma 1}, \alpha_{\sigma 2}) \cdots q_X(\alpha_{\sigma(2n-1)}, \alpha_{\sigma(2n)})$

Thm (Huybrechts) L: divisor on X: HK

$$\chi(L) \overset{HRR}{=\!=} \int_X td(X)\cdot \exp(c_1(L)) = \sum_{i=0}^{n} \int_X \boxed{\underline{td(X)_{2n-2i}}\,(L^{2i})} \cdot \frac{1}{(2i)!}$$

$$\underset{\text{Huybrechts}}{=\!=} \sum_{i=0}^{n} b_i \cdot q_X(L)^i$$

b_i independent of L.

this is called the RR polynomial of X.

Ex: X: K3. $\chi(L) = \frac{1}{2}L^2 + 2 = \frac{1}{2}q_X(L) + 2 \qquad q_X(L) = L^2$.

$X = S^{[n]}$ $\quad L$: div on S. $\rightsquigarrow L_n$ on $S^{[n]}$

$$\begin{tikzcd}
L \boxtimes L \boxtimes \cdots \boxtimes L \arrow[r] \arrow[d] & S \times S \times \cdots \times S \arrow[d, "/S_n"] & \\
\mathrm{Sym}^n L \arrow[r] & \mathrm{Sym}^n S & S^{[n]} \arrow[l, "\pi"']
\end{tikzcd}$$

$L_n := \pi^*(\mathrm{Sym}^n L)$ nef & big

$$\chi(L_n) \underset{\text{vanishing}}{=} h^0(L_n) = h^0(\mathrm{Sym}^n S, \mathrm{Sym}^n L)$$

$x_1^{n_1}, \dots, x_\ell^{n_\ell}$, $\sum_{i=1}^{\ell} n_i = n$.

$$= \binom{h^0(S,L)+n-1}{h^0(S,L)-1}$$

$$= \binom{h^0(S,L)+n-1}{n} = \binom{\frac{1}{2}L^2+n+1}{n}$$

$$\boxed{q(L_n) = L^2}$$

$$\parallel$$

$$\binom{\frac{1}{2}q_X(L)+n+1}{n}$$

$\rightsquigarrow$ $\forall M$: divisor on $X = K3^{[n]}$ & OG10

$$\chi(M) = \binom{\frac{1}{2}q_X(M)+n+1}{2}.$$

- $X = K_n(A)$, OG6 $\rightsquigarrow$ $\chi(M) = (n+1)\binom{\frac{1}{2}q_X(M)+n}{n}$

Thm (Jiang) $\forall X$: HK, $RR_X(q)$ has all positive coefficients.

$$\forall L,\ \chi(L) = \sum_{i=0}^{2n} b_i\, q_X(L)^i \qquad b_i > 0$$

Applications

Kawamata effective nonvanishing Conjecture: (open in dim ≥ 3)

$$\left\{\begin{array}{l} X: \text{sm projective variety} \\ L: \text{nef divisor s.t. } L - K_X: \text{nef \& big} \\ \quad (\text{ample}) \end{array}\right. \Rightarrow h^0(L) > 0.$$

Special case: $X : c_1(X)=0$, L: nef & big on X (ample)

$\Rightarrow h^0(L)>0$

(open in dim ≥ 5)

Cor: X: HK^{2n} L: nef & big on X

$\Rightarrow h^0(L) \geq n+2$.

$\left(\because h^0(L) \underset{\text{vanishing}}{=\!=} \chi(L) = \sum_{i=0}^{n} b_i q_X(L)^i > b_0 = \boxed{\chi(\mathcal{O}_X)=n+1} \right)$

Riess' theorem

Thm X: HK Assume $\boxed{\text{Conj}(\bigstar) \text{ holds}}$

L: nef & big divisor s.t. $Bs|L|$ contains a divisor

$\Rightarrow L = mE+F$ where

- $q_X(E)=0$, $m \geq 2$
- $q_X(E,F)>0$
- F: prime divisor with $q(F)<0$

in this case $\chi(L)=h^0(L)=\binom{m+n}{n}$

---

recall for K3.

Step 1 $L=M+F$. $\rightsquigarrow$ $M^2>0 \Rightarrow L^2=M^2, M.F=0 \overset{\text{HIT}}{\Rightarrow} F=0$ ↯

$\Rightarrow M^2=0$

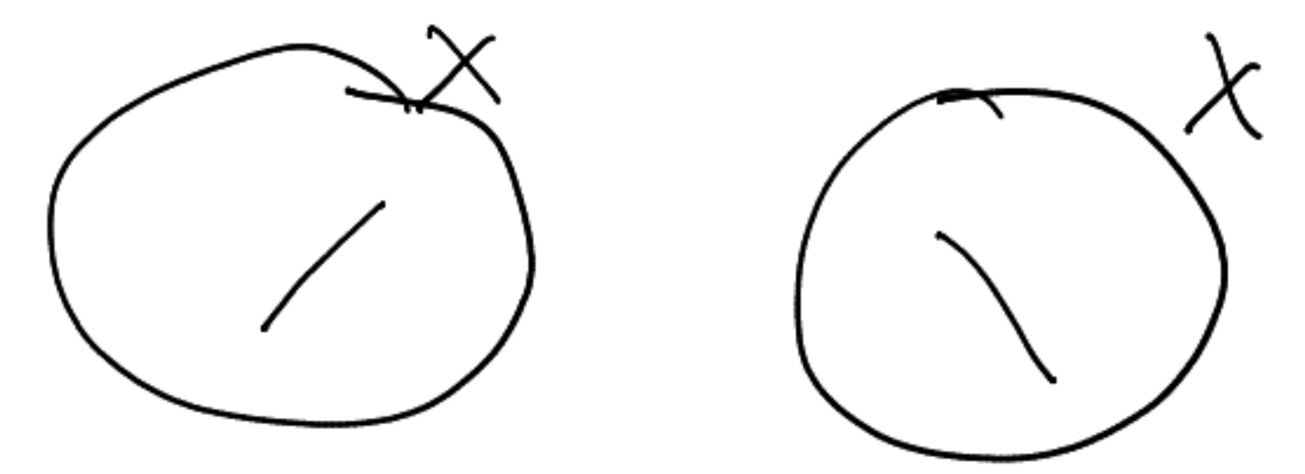
Step 2 $M^2=0 \Rightarrow M=mE$ $m\geq 2$

Step 3 $F=\sum a_i F_i$, $F_0.M>0 \Rightarrow L=\underbrace{(M+F_0)}_{\text{nef \& big}}+(F-F_0)$

$F_i \cong \mathbb{P}^1$ $\overset{\text{Step 1}}{\Rightarrow} F=F_0$.

MMP on HK

① if $X \overset{\text{birat}}{\underset{\phi}{\dashrightarrow}} X'$ birational HK



$\Rightarrow \begin{cases} (1) & \phi : \text{isomorphic in codim } 1 \\ (2) & \phi_* : H^2(X, \mathbb{R}) \xrightarrow{\cong} H^2(X', \mathbb{R}) \\ (3) & q_X = q_{X'} \end{cases}$

③ $N^1(X) = \{\text{divisors on } X\} / \equiv$ $\left(\begin{array}{l} D \equiv D' \Leftrightarrow D.C = D'.C \\ \qquad \forall C : \text{curve in } X \end{array} \right)$

free abelian group of finite rank.

$N^1(X)_{\mathbb{R}} = N^1(X) \otimes \mathbb{R}$ — $\mathbb{R}$-space.

$\begin{cases} D : \text{nef} \Leftrightarrow D.C \geq 0 \quad \forall C \subseteq X \text{ curve} \\ D : \text{movable} \Leftrightarrow |mD| \text{ has no fixed divisor} \end{cases}$

$\rightsquigarrow \begin{cases} \mathrm{Nef}(X) \subseteq N^1(X)_{\mathbb{R}} & \text{nef cone} \\ \mathrm{Mov}(X) \subseteq & \text{movable cone} \end{cases}$

- $\mathrm{Nef}(X)^\circ = \mathrm{Amp}(X)$: ample cone gen by ample divisors
- $\overline{\mathrm{Mov}}(X)$: closed movable cone

D : effective divisor on X

suppose D : not nef ($\Leftarrow \; \exists \, D.C < 0$)

$\Rightarrow \exists \; X \xrightarrow{f} Y$ birational morphism contracting some C with $C.D < 0$

Case 1 f : contracts a prime divisor (divisorial contraction)

Case 2 f : small.

in case 2.

$$\begin{tikzcd} X \arrow[rr, dashed, "\text{flop}", "\phi"'] \arrow[dr] & & \exists X' \arrow[dl] \\ D & Y & \phi_* D = D' \end{tikzcd}$$

X': HK.

In summary.

$$\begin{tikzcd}[column sep=small] X=X_1 \arrow[rr, dashed] \arrow[dr] & & X_2 \arrow[rr, dashed] \arrow[dl] \arrow[dr] & & X_3 \arrow[dl] & \cdots & X_n \arrow[dl] \arrow[dr] & \\ & Y_1 & & Y_2 & & Y_{n-1} & & Y_n \end{tikzcd}$$

① If $D \in \overline{\mathrm{Mov}}(X)$ & $q(X)>0$

$\Rightarrow \exists X \overset{\phi}{\dashrightarrow} X'$ s.t $\phi_* D$ nef & big

② If $D \notin \overline{\mathrm{Mov}}(X)$

$\Rightarrow \exists X \overset{\phi}{\dashrightarrow} X' \xrightarrow{f} Y$ s.t. f: divisorial contraction of $\phi_* D$

③ If $D \in \overline{\mathrm{Mov}}(X)$ & $q(X)=0$

Conjecture (☆) $\exists X \overset{\phi}{\dashrightarrow} \boxed{X' \xrightarrow{f} \mathbb{P}^n}$ ← Lagragian fibration.

s.t. $\phi_* D = f^* \mathcal{O}_{\mathbb{P}^n}(m) \quad m \geq 1$.

Lemma 1 if $D \in \overline{\mathrm{Mov}}(X)$ & $q(X)>0$

$\Rightarrow \chi(D) = h^0(D)$

Pf: $h^0(D) = h^0(X', \phi_* D) \overset{\text{vanishing}}{=\!=} \chi(X', \phi_* D)$

$= \sum_{i=0}^{n} b_i \, q_X(\phi_* D)^i = \chi(D)$

$q_X(\phi_* D)^i = q_X(D)^i$

Lemma 2 if $D \notin \overline{\mathrm{Mov}}(X)$, $X' \xrightarrow{f} Y$ contract F_0

$\left(\Rightarrow ① \ q_X(F_0) < 0, \ ② \ q_X(D, F_0) < 0 \ (\Rightarrow F_0 \subseteq \mathrm{Supp}\, D) \right)$

③ $q_{X'}(f^* G, F_0) = 0$

Pf: ③ WMA G: ample on Y.

$$q_{X'}(f^*G, F_0) \overset{?}{=} (f^*G)^{2n-1} \cdot F_0 = G^{2n-1} \cdot f_* F_0 = 0$$

$$c_X\, q_{X'}(mf^*G + F_0)^n \overset{BBF}{=} (mf^*G + F_0)^{2n}$$

$$\shortparallel$$

$$c_X\left(m^2 q_{X'}(f^*G) + 2m\, q_{X'}(f^*G, F_0) + q_{X'}(F_0)\right)^n$$

>0

compare $m^{2n-1} \rightsquigarrow$ $c_X \cdot \boxed{q_{X'}(f^*G)}^{n-1} \cdot 2q_{X'}(f^*G, F_0) \cdot n$

$= (mf^*G)^{2n-1} \cdot F_0 \cdot 2n.$

① $q_{X'}(f^*G) > 0$, $q_{X'}(f^*G, F_0) = 0 \Rightarrow q_{X'}(F) < 0$

③ $D' \equiv f^*(f_* D') + aF_0$

$$\begin{cases} D' \cdot C < 0 \\ F_0 \cdot C < 0 \end{cases}$$

$\Rightarrow a > 0$

$\Rightarrow q_{X'}(D', F_0) = a\, q_{X'}(F_0) < 0$

Lemm 3 in Lemma 2. if $D' = E' + F_0$ for some $E' \in \overline{\mathrm{Mov}}(X') \cap \mathrm{Pic}(X')$ with $q_X(E', F_0) > 0$

$\Rightarrow q_{X'}(2E' + F_0, F_0) > 0$

Pf: $D' = f^* f_* D' + aF_0$ $\Rightarrow a \leq 1$

$= E' + F_0$

Fact: (Namikawa)

$$\begin{array}{ccc} X'^{2n} & \xrightarrow{f} & Y^{2n} \\ F_0^{2n-1} & \longrightarrow & f(F_0) \end{array}$$

$(2n-2)$-dim.

$(xy = z^2)$

$\downarrow$

generically A_1, A_2-singularities

$\Rightarrow 2a \in \mathbb{Z}$ or $3a \in \mathbb{Z}$

$\Rightarrow a \in \{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}\}$

$\Rightarrow f^* f_* D' = E' + \{\frac{1}{2}, \frac{2}{3}, \frac{1}{3}\} F_0 \Rightarrow q_{X'}(2E' + F_0, F_0) > 0$

$\shortparallel$

$\circ E' + \bigcirc$

pf of Riess thm. X: HK. Conj ($\spadesuit$) holds

$L = M + F \quad F \neq 0 \quad L$: nef big

$\Rightarrow \begin{cases} \circ\ L = mE + F,\ m \geq 2 \\ \circ\ E: \text{movable } q_X(E) = 0,\ h^0(E) = n+1. \\ \circ\ q(E, F) > 0 \\ \circ\ F: \text{prime. } q(F) < 0 \end{cases}$

Step 1 Suppose $L = M' + F'$ with $0 \leq F' < F$

$\begin{cases} \circ\ M' \in \overline{\mathrm{Mov}}(X) \cap \mathrm{Pic}(X) \\ \circ\ q(M') > 0 \end{cases}$

then $F' = 0$

pf:
$$\begin{array}{ccc} h^0(L) & = & h^0(M') \\ \text{KV}\ \| & & \|\ \text{Lem 1} \\ \chi(L) & \cdots\cdots & \chi(M') \end{array}$$

$\Rightarrow q(L) = q(M') \quad (\chi(-) = \sum b_i q(-)^i \text{ increasing })$

$$\begin{aligned} &\| \\ &q(M'+F') = q(M') + \underline{2q(M', F') + q(F')} \end{aligned}$$

$\Rightarrow \underset{\underbrace{q(L,F')}_{\geq 0} + \underbrace{q(M',F')}_{\geq 0}}{\underset{\|}{2q(M', F')}} + q(F') = 0 \Rightarrow \begin{cases} q(M', F') = 0 \\ q(F') = 0 \end{cases} \Rightarrow \underline{F' = 0}$

Step 2 $L = M + F. \xRightarrow[\text{Step 1}]{} q(M) = 0 \xRightarrow[\substack{\text{MMP} \\ \text{Conj}(\spadesuit)}]{}$
$$\begin{array}{ccc} X & \overset{\phi}{\dashrightarrow} & X' \\ & & \downarrow f \\ & & \mathbb{P}^n \end{array}$$

$M = \phi^* f^* \mathcal{O}_{\mathbb{P}^n}(m) = m E \quad E := \underline{\phi^* f^* \mathcal{O}_{\mathbb{P}^n}(1)}$

$h^0(E) = n+1 \quad h^0(M) = h^0(L) \underline{> n+1} \Rightarrow m \geq 2.$

$\rightsquigarrow L = mE + F \qquad E \in \overline{\mathrm{Mov}}(X) \cap \mathrm{Pic}(X)$

$q(E) = 0$

<u>Step3</u> F: prime

$F = \sum a_i F_i \qquad q_X(L, E) > 0$

$\quad \| $

$q_X(F, E)$

$\Rightarrow \exists F_0 \subseteq F,\ q_X(E, F) > 0$

$\rightsquigarrow L = (mE + F_0) + F - F_0$

<u>Claim1</u> $mE + F_0 \in \overline{\mathrm{Mov}}(X)$

<u>Claim2</u> $q_X(mE + F_0) > 0$ $\xrightarrow{\text{Step1}} F - F_0 = 0$

<u>Claim1</u> $mE + F_0 \notin \overline{\mathrm{Mov}}(X) \xRightarrow{\text{Lem2}} X \overset{f}{\dashrightarrow} X' \rightarrow Y$ $\supseteq F_1$

$\Rightarrow E + F_0 \notin$

contract $F_1 \subseteq \mathrm{Supp}(\underline{mE'} + F_0')$

$\Rightarrow F_1 = F_0'$

$q(mE + F_0, F_0) < 0$

$q(E + F_0, F_0) < 0 \underset{\text{Lem3}}{\Rightarrow} q(2E + F_0, F_0) > 0$ $\lightning$

$\Rightarrow mE + F_0 \in \overline{\mathrm{Mov}}(X)$.

<u>Claim2</u>. if $E + F_0 \in \overline{\mathrm{Mov}}(X) \Rightarrow q(mE + F_0) > 0$

$= \underbrace{q(mE, mE + F_0)}_{\geq 0} + q(F_0, mE + F_0)$

$\| $

$\underset{\geq 0}{q(F, E+F)} + \underset{> 0}{q(F, (m-1)E)}$

- if $E + F_0 \notin \overline{\mathrm{Mov}}(X)$

$\Rightarrow q(E + F_0, F_0) > 0 \Rightarrow q(mE + F_0) > 0$.

$\square$

3. Day 3& 4

Exercise 3.1. Let X be a hyperkähler manifold of dimension $2n$ and q_X be the BBF form. Show that for any $\alpha_1, \ldots, \alpha_{2n} \in H^2(X, \mathbb{R})$,

$$\int_X \alpha_1 \ldots \alpha_{2n} = \frac{c_X}{(2n)!} \sum_{\sigma \in S_{2n}} q_X(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}) \ldots q_X(\alpha_{\sigma(2n-1)}, \alpha_{\sigma(2n)})$$

Exercise 3.2. Let X be a hyperkähler manifold of dimension $2n$ and q_X be the BBF form. Show that for any prime divisors $F_1 \neq F_2$, $q_X(F_1, F_2) \geq 0$. (Hint: compare with $F_1 \cdot F_2 \cdot H^{2n-2}$ for some ample divisor H.)

Exercise 3.3. Let $f : X \to Y$ be a projective birational morphism between normal varieties. Let E be an effective exceptional divisor. Show that there exists a curve C on X mapping to a point of Y such that $E \cdot C < 0$