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Positivity of the cotangent bundle

Talk III

Today: geometric context

Foliations

Defn: X normal variety, let $T_X := \Omega_X^*$ the tangent sheaf

A foliation is a saturated subsheaf $\mathcal{F} \subset T_X$ that is integrable, i.e. closed under the Lie bracket.

We call $K_{\mathcal{F}} := \det \mathcal{F}^*$ the canonical divisor of $\mathcal{F}$.

NB: The Lie bracket $[\,,\,] : T_X \times T_X \to T_X$ is not $\mathcal{O}_X$-linear but it induces an $\mathcal{O}_X$-linear map

$$\Lambda^2 \mathcal{F} \to T_X/\mathcal{F}$$

that is zero iff $\mathcal{F}$ is integrable.

Ex: If $\varphi : X \to Y$ fibration onto variety Y, we have exact sequence

$$\varphi^*\Omega_Y \to \Omega_X \to \Omega_{X/Y} \to 0.$$

Dualise $\Omega_{X/Y}^* \to \Omega_X^* = T_X$ and saturate the image to $\mathcal{F} \subset T_X$ the foliation induced by φ.

If φ is smooth, then $\mathcal{F} = T_{X/Y}$

Frobenius' thm:

X a manifold, $\mathcal{F} \subset T_X$ a regular foliation, that is $\mathcal{F} \subset T_X$ is a subbundle.

Then for every $x \in X$ there exists an analytic neighbourhood $\underset{\overset{\cup}{x}}{U} \subset X$ and holomorphic submersion

$$\varphi : U \to Y \quad \text{s.t.} \quad \mathcal{F}|_U = T_{U/Y}.$$

Globally a regular foliation defines an equivalence relation on X: X a manifold, then two points $x_1, x_2 \in X$ are equivalent if $\exists$ connected chain of curves $(C_i) \subset X$ $i = 1..d$

open, analytic

s.t. $T_{C_i} \subset \mathcal{F}|_{C_i}$

and $x_1 \in C_0$ and $x_2 \in C_\alpha$

An equivalence class of this relation is called a leaf of the foliation.

NB: leaves of a foliation are locally closed subsets of X, but in general they are not closed submanifolds.

$X = \mathbb{C}^2 / \Lambda$ complex torus / abelian surface

A linear subspace $L \subset \mathbb{C}^2$ defines a regular foliation $\mathcal{F} \subset T_X$ but for a sufficiently general choice the leaves of the foliation are dense.

Theorem (Bogomolov-McQuillan 2001, Kebekus-Sola Condé-Toma 2006)

X projective manifold, $\mathcal{F} \subset T_X$ a foliation.

Let $C \subset X$ be a (projective) curve s.t.

- $\mathcal{F}$ is regular in a neighbourhood of C
- $\mathcal{F}|_C$ is an ample vector bundle.

Let $x \in C$ be a general point and $L_x \subset X$ be the leaf passing through x.

Then $\overline{L_x}^{Zar}$ has dimension $= \mathrm{rk}\, \mathcal{F}$ (L_x is quasi-projective and $L_x \subset \overline{L_x}^{Zar}$ Zariski open)

and is covered by rational curves $\mathbb{P}^1 \to X$ (actually the leaf is rationally connected)

Miyaoka's theorem $\qquad \dim X = n$

X projective manifold. Assume that there exists $H \to X$ ample s.t. $K_X \cdot H^{n-1} < 0$

Then X is covered by rational curves.

Proof:

Warm-up: Assume T_X is H-sstable

Let $C \subset X$ be a MR-curve, i.e. $C = D_1 \cap \dots \cap D_{n-1}$

$D_i \in |mH|$

and $T_X|_C$ semistable.

Miyaoka's lemma: $(T_X \otimes \frac{1}{n} K_X)|_C$ nef

$$\Rightarrow T_X|_C = \underbrace{(T_X \otimes \tfrac{1}{n} K_X)}_{\text{nef}}|_C \otimes \underbrace{\tfrac{1}{n} K_X^*|_C}_{\text{ample}} \text{ ample.}$$

Now apply Bogomolov-McQuillan for $\mathcal{F} = T_X$.

General case: consider HN filtration

$$0 \subset HN_1(T_X) \subset \dots \subset HN_\ell(T_X) = T_X$$

and set $\mathcal{F} := HN_1(T_X) \subset T_X$ saturated subsheaf

Is $\mathcal{F} \subset T_X$ integrable? Have to check that morphism

$$\underline{\Lambda^2 \mathcal{F}} \to \underline{T_X/\mathcal{F}} \quad \text{is zero}$$

$T_X/\mathcal{F}$ has filtration s.t. graded pieces are

$$Gr_2(T_X), \dots, Gr_\ell(T_X)$$

In particular slopes are $< \mu(Gr_1(T_X)) = \mu(\mathcal{F})$.

Easy computation $\mu_H(\Lambda^2 \mathcal{F}) = 2\mu_H(\mathcal{F}) > \mu_H(\mathcal{F})$

Note $\mu_H(T_X) > 0$ so $\mu_H(\mathcal{F}) > 0$.

$$\overset{\text{exercise}}{\Longrightarrow} \quad H^0(X, \mathcal{H}om(\Lambda^2 \mathcal{F}, T_X/\mathcal{F})) = 0.$$

hence $\mathcal{F}$ is integrable.

Now if $C \subset X$ is MR-curve, then

- $\mathcal{F}$ is regular near C (exercise from sheet)
- $\mathcal{F}|_C$ is ample ($\mathcal{F}|_C$ sstable + $\mu_H(\mathcal{F}) > 0$)

Bogomolov-
$\Rightarrow$
McQuillan
general $\mathcal{F}$-leaf is algebraic and the closure is covered by rational curves.

$\square$

Disadvantages of Bogomolov-McQuillan:

- apart from MR-curves it is not easy to check that $\mathcal{F}|_C$ is ample.
- If $\mathcal{F}|_C$ is ample, then $K_{\mathcal{F}} \cdot C = \det \mathcal{F}^* \cdot C < 0$

Thus statement never applies to foliations where $K_{\mathcal{F}}$ is nef.

Defn: X normal projective variety, $\mathcal{F} \subset T_X$ foliation is algebraically integrable if for a general point $x \in X$ the leaf L_x is quasi-projective

$$\downarrow$$

$\overline{L_x}^{Zar}$ has dimension $= \operatorname{rk} \mathcal{F}$.

Remark: $\mathcal{F}$ is algebraically integrable

$$\Updownarrow$$

$\exists\, \varphi : X \dashrightarrow Y$ rational map s.t. $\mathcal{F}$ is induced by φ.

Thm (Bost, Campana-Păun, Druel)

X normal projective variety, $\mathcal{F} \subset T_X$ foliation.
If $\mathcal{F}^*$ is not pseudoeffective, then $\mathcal{F}$ is algebraically integrable.

Thm (H - Peternell 2017)

X normal projective variety of dimension n that is smooth in codimension two, i.e.

$$\operatorname{codim} X_{sing} \geq 3$$

Fix $H \to X$ ample divisor. Let $\mathcal{E} \to X$ reflexive sheaf s.t.

1) $c_1(\mathcal{E}) . H^{n-1} = 0$

2) $S^{[l]}\mathcal{E}$ is H-stable for all $l \in \mathbb{N}$.

3) $\mathcal{E}$ is pseff.

Then $c_2(\mathcal{E}) . H^{n-2} = 0$.

Cor: X proj. manifold, $\mathcal{F} \subset T_X$ foliation s.t.

$S^{[l]}\mathcal{F}$ is H-stable $\forall l \in \mathbb{N}$

if $\underline{c_1(\mathcal{F}) = 0,\ c_2(\mathcal{F}) \neq 0}$ then

$\mathcal{F}$ is algebr. integrable.

Cor: X projective manifold s.t. $K_X \simeq \mathcal{O}_X$ and $\pi_1(X) = \{1\}$

Then Ω_X and T_X are not pseff.

Focus on easiest case:

Prop: X a K3 surface, then Ω_X is not pseff.

(Nakayama, BDPP)
2004

Preliminary results: X K3 surface

$\overset{\text{Yau}}{\Longrightarrow}$ $S^l\Omega_X$ stable for all $l \in \mathbb{N}$. and every $H \to X$ ample.

We also know:
- $c_1(X) = 0$ by defn.
- $c_2(X) = 24$ by Noether's formula
- Ω_X is not nef:

If $V \to$ Surface nef vector bundle, then

$$c_1^2(V) \geq c_2(V) \geq 0$$

In our case $c_1 = 0$, so we would get $c_2 = 0$.

Arguing by contradiction we assume

$V := \Omega_X$ pseff

and will deduce $\underset{\substack{\| \\ : \\ V}}{\Omega_X}$ nef $\lightning$

$$\begin{tikzcd} \mathbb{P}(V) \arrow[d, "\pi"] & \xi_V \arrow[l] \\ X & \end{tikzcd}$$

ξ_V is not nef, so $\exists B \subset \mathbb{P}(V)$ irreducible curve s.t. $\xi_V \cdot B < 0$

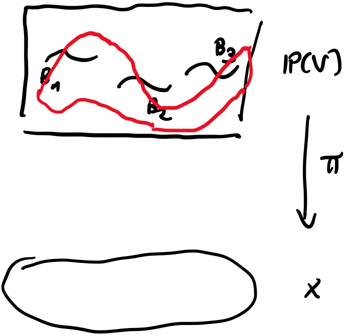
Two cases:

① There are only countably many curves $B_i \subset \mathbb{P}(V)$ s.t. $\xi_V \cdot B_i < 0$

② There are uncountably many curves B s.t. $\xi_V \cdot B < 0$

$\Leftrightarrow$ $\exists Z \subset \mathbb{P}(V)$ irreducible surface that is covered by such curves.

Case 1



If $D \subset \mathbb{P}(V)$ is a very general very ample divisor, then

$$B_i \not\subset D \quad \forall i.$$

$$\Rightarrow \xi_V|_D \text{ is nef}$$

As in Druel's proof choose $H \to X$ ample and $m \gg 0$ s.t.

$V \otimes H^{\otimes m}$ is very ample $\Leftrightarrow$ $\mathcal{O}_{\mathbb{P}(V)}(1) \otimes \pi^* H^{\otimes m}$ is very ample.

We can choose $D \in |\mathcal{O}_{\mathbb{P}(V)}(1) \otimes \pi^* H^{\otimes m}|$ s.t.

$\xi_V|_D$ is nef.

In particular

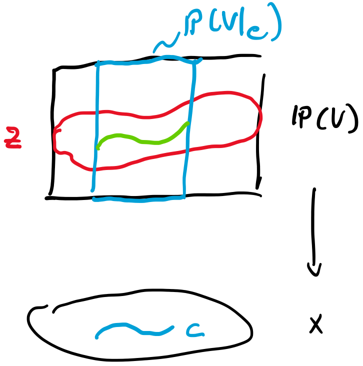
$$0 \leq (\xi_V|_D)^2 = \xi_V^2 \cdot (\xi_V + m\,\pi^* H)$$
$$= \xi_V^3 + m\,\xi_V^2\,\pi^* H = *$$

Grothendieck relation $\xi_V^2 = \pi^* c_1(V)\,\xi_V - \pi^* c_2(V)$

$$* = -c_2(V) \qquad \Rightarrow c_2(V) \leq 0$$

but we know $c_2(V)=24$ $\lightning$

Case 2:



$Z \subset \mathbb{P}(V)$ irred. surface covered by curves B s.t. $\xi_V . B<0$.

ξ_V is pseff, then Boucksom-Nakayama

$\exists\, a>0$ s.t. $\xi_V - aZ$ is pseff divisor.

Choose now $C \subset X$ a MR-curve s.t.

$S^l V|_C$ stable $\forall\, l \in \mathbb{N}$

and $(\xi_V - aZ)|_{\mathbb{P}(V_C)}$ is pseff.

Since $V|_C$ sstable and $c_1(V|_C)=0$ we know by Miyaoka's lemma $V|_C$ is nef

$$\iff \xi_V|_{\mathbb{P}(V_C)} \text{ is nef}$$

$c_1(V|_C)=0$

$$\implies (\xi_V|_{\mathbb{P}(V_C)})^2 = 0 \qquad (1)$$

$$\underbrace{\xi_V|_{\mathbb{P}(V_C)}}_{\text{nef}} \underbrace{(\xi_V - aZ)}_{\text{psef}}|_{\mathbb{P}(V_C)} \geq 0 \qquad (2)$$

$$\xi_V|_{\mathbb{P}(V_C)} \cdot \underset{Z \cap \mathbb{P}(V_C)}{\overset{Z|_{\mathbb{P}(V_C)}}{\|}} \geq 0 \qquad (3)$$

green curve from picture.

1+2+3

$$\implies \xi_V|_{\mathbb{P}(V_C)} \cdot Z|_{\mathbb{P}(V_C)} = 0.$$

Thus we have found a curve $\subset \mathbb{P}(V_C)$ that is orthogonal to $\xi_V|_{\mathbb{P}(V_C)}$. ↯ contradiction ↯ to Mumford

Lemma (Mumford's example)

C smooth proj. curve, $V \to C$ vector bundle s.t. $S^l V$ stable $\forall l \in \mathbb{N}$.

If $\operatorname{rk} V = 2$, $c_1(V) = 0$ then

$\xi . C' > 0$ for every curve $C' \subset \mathbb{P}(V)$.