Introduction to hyperbolicity
Lecture 4 : Hyperbolicity of hypersurfaces

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Plan

1. Kobayashi conjecture
2. Invariant jet differentials
3. Wronskians approach
4. Variational method

The Kobayashi conjecture

Kobayashi conjectured the following.

Conjecture

Let H be a general hypersurface in $\mathbb{P}^n$ of degree $d \geqslant 2n+1$, then

1. *H is hyperbolic.*
2. *$\mathbb{P}^n \setminus H$ is hyperbolically embedded in $\mathbb{P}^n$.*

Here general means that there exists a non-empty Zariski open subset

$$U \subset \mathbb{P}(H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d)))$$

such that the conclusion is satisfied for all $H \in U$.

Remarks on the conjecture

1. If $n = 2$, then the compact case (1) is obviously true since H is a curve and $g(H) = \frac{(d-1)(d-2)}{2}$.
2. If $H \subset \mathbb{P}^n$ is a hypersurface of degree d, then

$$K_H = \mathscr{O}_H(d - N - 1).$$

   In particular, if GGL is true then, if $d \geqslant N + 2$, any smooth hypersurface wouldn't contain a Zarski dense entire curve.
3. If $d \leqslant 2n - 3$ then any H contains a line, in particular, it is not hyperbolic.
4. For any d, there exists a smooth hypersurface of degree d containing a line, in particular, the genericity assumption is essential.

Algebraic side

1. Voisin, generalizing works of Clemens and Ein, proved that if $d \geqslant 2n$, then very general hypersurfaces of degree d are algebraically hyperbolic and that every subvariety is of general type.
2. Pacienza improved the bound for $n \geqslant 6$ and proved that every subvariety of a very general hypersurfaces of degree $d \geqslant 2n - 2$ is of general type.
3. In the logarithmic setting, similar results were obtained by Pacienza and Rousseau.

Known results

If one doesn't care about the bound on the degree in the conjecture, then the Kobayashi conjecture has now been proven.

Theorem (Siu, B., Deng, Demailly, Diverio-Merker-Rousseau, Riedl-Yang, Berczi-Kirwan...)

For any $N \geqslant 2$, there exists $d_N \in \mathbb{N}$ such that for any $d \geqslant d_N$, a general hypersurface $H \subset \mathbb{P}^n$ of degree d is hyperbolic and its complement is hyperbolically embedded in $\mathbb{P}^n$.

The number d_N is explicite, but for now, not optimal.

Strategies

There are now two types of proof.

1. Wronskian approach : B., Deng, Demailly. Best bound $n^{2n+3}(n+1)$.
2. Variational method : Voisin, Siu, Diverio-Merker-Rousseau, Riedl-Yang, Berczi-Kirwan... Latest developments yields the bound $(2n-3)^6$ (in the compact case).

We will for now on focus only on the compact part of the conjecture, namely the hyperbolicity of hypersurfaces, not their complements. The complement case can be treated similarly using logarithmic jet differentials instead. Also, for simplicity, we will not discuss the bounds in details.

Invariant jet differentials

Let us first refine a bit the theory of jet differentials introduced last time. Let X be an n-dimensional complex manifold. Let $p_k : J_kX \to X$ be its k-th order jet space.
Consider the group of k jets of biholomorphism

$$\mathbb{G}_k = \left\{ \varphi : t \mapsto a_1 t + a_2 t^2 + \cdots + a_k t^k \mid a_1 \in \mathbb{C}^*, a_j \in \mathbb{C} \ \forall j \geqslant 2 \right\} / t^{k+1}.$$

This group acts on J_kX by

$$\varphi \cdot j_k f = j_k(f \circ \varphi).$$

And we can now look at jet differentials of a given weight with respect to this action.

Invariant jet differentials

For any $k, m \in \mathbb{N}$ consider the (locally free) subsheaf $E_{k,m}\Omega_X \subset E^{\mathrm{GG}}_{k,m}\Omega_X$ defined by

$$E_{k,m}\Omega_X(U) = \left\{ \begin{array}{c} \omega \in \mathscr{O}\left(p_k^{-1}(U)\right) \mid \omega(\varphi \cdot j_k\gamma) = \varphi'(0)^m \omega(j_k\gamma), \\ \forall \varphi \in \mathbb{G}_k \ \forall j_k\gamma \in p_k^{-1}(U) \end{array} \right\}.$$

Why consider invariant jets differential?

It is natural to study invariant jet differentials first of all because if one studies hyperbolicity, one is only concerned with the image of entire curves, not the way they are parametrized, so it is natural to take this into account.
Also, in general the bundles $E_{k,m}\Omega_X$ are expected to have better positivity properties than the bundles $E_{k,m}^{\mathrm{GG}}\Omega_X$.
Lastly, there is an interesting geometry behind these differentials, based on the Demailly-Semple jet tower that we will now describe.

The Demailly-Semple jet tower

Let X be a complex manifold of dimension n. Let $V \subset T_X$ be a rank r sub vector bundle. (We say that (X, V) is a directed manifold).
Set

$$\pi : \widetilde{X} := P(V) \to X.$$

Let $\mathscr{O}_{\widetilde{X}}(-1)$ be the tautological line bundle on $\widetilde{X}$, by definition, one has, for any $(x, [\xi]) \in \widetilde{X}$ $(x \in X,\ \xi \in V_x \setminus \{0\})$,

$$\mathscr{O}_{\widetilde{X}}(-1)(x, [\xi]) = \mathbb{C}\xi.$$

This is a subbundle of $\pi^* V \subseteq \pi^* T_X$.

Demailly-Semple jet tower

We can define a rank r vector bundle $\widetilde{V} \subset T\widetilde{X}$ as follows

$$\widetilde{V}_{(x,[\xi])} := \left\{ \eta \in T_{(x,[\xi])}\widetilde{X} \ ; \ \ \pi_* \eta \in \mathbb{C}\xi \right\}.$$

That is to say, it is the vector bundle sitting in the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_{\widetilde{X}/X} & \longrightarrow & T_{\widetilde{X}} & \longrightarrow & \pi^* T_X & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & T_{\widetilde{X}/X} & \longrightarrow & \widetilde{V} & \longrightarrow & \mathscr{O}_{\widetilde{X}}(-1) & \longrightarrow & 0
\end{array}$$

Therefore, we get a new directed manifold $(\widetilde{X}, \widetilde{V})$.

Demailly-Semple jet tower

A key point of this construction is the lifting property. Let (X, V) as before. Take C be any Riemann surface (non necessarily compact) and take a non-constant morphism $f : C \to X$ which is tangent to V (i.e. $f'(t) \in V$ for all $t \in C$). Then by differentiating f, we obtain a map

$$\tilde{f} : C \to \widetilde{X}$$

which is tangent to $\widetilde{V}$ and such that

$$\pi \circ \tilde{f} = f.$$

Demailly-Semple jet tower

Now we can construct the Demailly-Semple jet tower inductively using the previous construction.

1. Set $(X_0, V_0) = (X, T_X)$..
2. Set $(X_1, V_1) = (\widetilde{X_0}, \widetilde{V_0})$
3. For any $k \geqslant 2$, set $(X_k, V_k) = (\widetilde{X}_{k-1}, \widetilde{V}_{k-1})$.

For any k, this construction yields a tautological bundle $\mathscr{O}_{X_k}(1)$ on X_k and a map

$$\pi_{k,k-1} : X_k \to X_{k-1}.$$

Consider also the compostion $\pi_k : X_k \to X$.
For any Riemann surface C, any morphism $f : C \to X$ is canonically lifted to a morphism

$$f_{[k]} : C \to X_k$$

such that $\pi_k \circ f_{[k]} = f$.

Demailly-Semple jet tower

This construction satisfies the following theorem.

Theorem (Demailly)

With the above notation. Let

$$J_kX^{\mathrm{reg}} = \left\{ j_kf \in J_kX \ ; \ f'(0) \neq 0 \right\}.$$

There exists a divisor $X_k^{\mathrm{sing}} \subset X_k$, such that the complement X_k^{reg} is isomorphic to

$$J_kX^{\mathrm{reg}}/\mathbb{G}_k.$$

Therefore X_k is a compactification of the quotient $J_kX^{\mathrm{reg}}/\mathbb{G}_k$.

Let us mention that $\mathbb{G}_k$ is not reductive, therefore the existence of a quotient is not immediate.

Singular jets

The locus of singular jets X_k^{sing} can be understood very explicitly. Indeed, for any $k \geqslant 2$ one has an exact sequence

$$0 \to T_{X_{k-1}/X_{k-2}} \to V_{k-1} \to \pi_{k-1,k-2}^{*} \mathscr{O}_{X_{k-2}}(-1) \to 0$$

therefore, $X_k = P(V_{k-1})$ contains a divisor

$$D_k = P(T_{X_{k-1}/X_{k-2}}).$$

By construction, this divisor belongs to

$$|\mathscr{O}_{X_k}(1) \otimes \mathscr{O}_{X_{k-1}}(-1)|.$$

With this notation

$$X_k^{\mathrm{sing}} = \bigcup_{j=2}^{k} \pi_{k,j}^{-1} D_j.$$

Demailly-Semple jet tower

Crucial to us is the following theorem.

Theorem (Demailly)

Notation as before. For any $m \in \mathbb{N}$, one has

$$(\pi_k)_* \mathscr{O}_{X_k}(m) = E_{k,m}\Omega_X.$$

In particular, denoting for any $a_1, \ldots, a_k \in \mathbb{Z}$

$$\mathscr{O}_{X_k}(a_1, \ldots, a_k) = \mathscr{O}_{X_1}(a_1) \otimes \cdots \otimes \mathscr{O}_{X_k}(a_k)$$

the fundamental vanishing theorem, yields,

Demailly-Semple jet tower

Theorem (Demailly)

For any $a_1, \ldots, a_k \in \mathbb{N}$, *for any ample line bundle* A *on* X, *and any entire curve* $f : \mathbb{C} \to X$. *For any*

$$\omega \in H^0(X_k, \mathscr{O}_{X_k}(a_1, \ldots, a_k) \otimes \pi_k^* A^{-1})$$

one has

$$f_{[k]}^* \omega = 0,$$

Or equivalently, $f_{[k]}(\mathbb{C}) \subset (\omega = 0)$. *In particular*

$$f_{[k]}(\mathbb{C}) \subset \mathbb{B}_+(\mathscr{O}_{X_k}(a_1, \ldots, a_k)).$$

Demailly-Semple jet tower

Therefore, we are reduced to study the positivity of the bundles $\mathscr{O}_{X_k}(a_1, \ldots, a_k)$. One can even go one step further. Suppose $\mathcal{J} \subset \mathscr{O}_{X_k}$ is an ideal sheaf supported on X_k^{sing}. Let

$$\nu_k : \widehat{X}_k \to X_k$$

is a resolution of this ideal which is an isomorphism on X_k^{sing} and write

$$\mathscr{O}_{\widehat{X}_k}(-E) = \nu_k^{-1}\mathcal{J}.$$

Then every entire curve $f : \mathbb{C} \to X$ lifts to a curve

$$\hat{f}_{[k]} : \mathbb{C} \to \widehat{X}_k$$

such that

$$\hat{f}_{[k]}(\mathbb{C}) \subset \mathbb{B}_+(\nu_k^*\mathscr{O}_{X_k}(a_1, \ldots, a_k) \otimes \mathscr{O}_{\widehat{X}_k}(-E))$$

Wronskian approach

We will now give some ideas of the proof of the Kobayashi conjecture relying on Wronskians

Local Wronskians

We start by a construction of generalized Wronskians.
Given any open subset $U \subset X$ and any holomorphic functions $f_0, \ldots, f_k \in \mathscr{O}(U)$, one can consider the Wronskian

$$W(f_0, \ldots, f_k) = \begin{vmatrix} f_0 & f_1 & \cdots & f_k \\ df_0 & df_1 & \cdots & df_k \\ \vdots & \vdots & \ddots & \vdots \\ d^k f_0 & d^k f_1 & \cdots & d^k f_k \end{vmatrix}$$

By an explicite computation, one shows that this is an element of

$$E_{k,k'}\Omega_X(U)$$

where $k' = \frac{k(k+1)}{2}$.

Global Wronskian

This construction can be globalized in the following way. Let L be a line bundle on X, for any

$$\sigma_0, \dots, \sigma_k \in H^0(X, L)$$

on can set, for any local representative $f_1, \dots, f_k$ of $\sigma_1, \dots, \sigma_k$ over some trivializing open subset $U \subset X$

$$W(\sigma_0, \dots, \sigma_k) = \begin{vmatrix} f_0 & f_1 & \cdots & f_k \\ df_0 & df_1 & \cdots & df_k \\ \vdots & \vdots & \ddots & \vdots \\ d^k f_0 & d^k f_1 & \cdots & d^k f_k \end{vmatrix}.$$

One then verifies that this defines an element in

$$H^0(X, E_{k,k'}\Omega_X \otimes L^{k+1}) \cong H^0(X_k, \mathscr{O}_{X_k}(m) \otimes \pi_k^* L^{k+1}).$$

Wronskian ideal sheaf

By multilinearity we get a map

$$\Lambda^{k+1} H^0(X, L) \to \mathscr{O}_{X_k}(m) \otimes \pi_k^* L^{k+1}$$

Whose image defines an ideal sheaf $\mathfrak{w}_k \subset \mathscr{O}_{X_k}$ One verifies that this ideal sheaf is independent of the choice of L as soon as L separates k-jets at every point (e.g. $L = A^k$ for a very ample line bundle A on X.) One also verifies that under this assumption, the ideal sheaf $\mathfrak{w}_k$ is supported on X_k^{sing}.

In particular, in view of what we said before, if $\mu_k : \widehat{X}_k \to X_k$ is a suitable resolution of $\mathfrak{w}_k$ and $\mathscr{O}_{\widehat{X}_k}(-E) = \mu_k^{-1}\mathfrak{w}_k$ then we can understand the entire curves in X by studying the positivity of the line bundle

$$\mu_k^* \mathscr{O}_{X_k}(a_1, \ldots, a_k) \otimes \mathscr{O}_{\widehat{X}_k}(-E).$$

Wronskian ideal sheaf

In fact, one can show that this wronskian construction has several functorial property, in particular in can be performed in families, and one can also chose a resolution that can be made in families. The upshot is the following proposition

Proposition

Let $\rho : \mathscr{X} \to B$ be a smooth family of projective variety. Suppose that there is a fiber $X = X_0$ of ρ and integers $a_1, \ldots, a_k$ such that the bundle

$$\mu_k^* \mathscr{O}_{X_k}(a_1, \ldots, a_k) \otimes \mathscr{O}_{\widehat{X}_k}(-E)$$

is ample, then the same property holds for every fiber X_t for t in a non-empty Zariski open subset $U \subset B$. And in particular, for any $t \in U$, X_t is hyperbolic.

Idea of proof of the Kobayashi conjecture

Therefore, to prove the Kobayashi conjecture, one only needs to construct a single example of a hypersurface satisfying the strong positivity property

$$\mu_k^* \mathscr{O}_{X_k}(a_1, \ldots, a_k) \otimes \mathscr{O}_{\widehat{X}_k}(-E)$$

ample for some k and some $a_1, \ldots, a_k$.

Fermat hypersurface

The idea is to construct such an example by deforming slightly a Fermat type hypersurface. To get a feeling why this might be true, consider the Fermat hypersurface of degree d, $H \subset \mathbb{P}^n$ given by

$$z_0^d + \cdots + z_n^d = 0.$$

Let $k = n - 1 = \dim H$. Consider the Wronskian

$$W(z_1^d, \ldots, z_n^d) \in H^0(H, E_{k,k'}\Omega_H \otimes \mathscr{O}_H(dn)).$$

Here we used $k + 1 = n$, and $z_i^d \in H^0(H, \mathscr{O}_H(d))$.

Fermat hypersurface

The point is that if one does the computation, one observes that

$$W(z_1^d,\dots,z_n^d)=\begin{vmatrix} z_1^d & \cdots & z_n^d \\ \vdots & \ddots & \vdots \\ d^k z_1^d & \cdots & d^k z_n^d \end{vmatrix}=\begin{vmatrix} z_1^{d-k}z_1^k & \cdots & z_n^{d-k}z_n^k \\ z_1^{d-k}g_1^1 & \cdots & z_n^{d-k}g_n^1 \\ \vdots & \ddots & \vdots \\ z_1^{d-k}g_1^k & \cdots & z_n^{d-k}g_n^k \end{vmatrix}.$$

For some (locally defined) jet differential forms g_i^j. Therefore $W(z_1^d,\dots,z_n^d)$ is divisible by $z_1^{d-k}\cdots z_n^{d-k}$.

Fermat hypersurface

Using the fact that we are on the Fermat hypersurface, we can moreover use the relation

$$z_0^d + \cdots + z_n^d = 0$$

and its derivatives, in order to obtain $W(z_1^d, \ldots, z_n^d) = -W(z_0^d, z_2^d, \ldots, z_n^d)$ and the same argument proves that $W(z_1^d, \ldots, z_n^d)$ is also divisible by z_0^{d-k}. Altogether, we get an element

$$\frac{W(z_1^d, \ldots, z_n^d)}{z_0^{d-k} \cdots z_n^{d-k}} \in H^0(H, E_{k,k'}\Omega_H \otimes \mathscr{O}_H(dn - (n+1)(d-k))).$$

But observe that if $d \geqslant n^2$, then the twist

$$dn-(n+1)(d-k)) = dn-(n+1)d+(n+1)(n-1) = -d+n^2-1 < 0.$$

Fermat hypersurface

Therefore, we have one jet differential (which actually belongs to the Wronskian ideal sheaf if seen as a global section of $\mathscr{O}_{H_k}(k'))$ that can be used to control entire curves. It is of course not enough, since Fermat hypersurfaces always contain entire curves because they contain lines. But if one deforms suitably the Fermat hypersurfaces, similar arguments can produce many more such Wronskian jet differential equations, enough to actually prove the ampleness of $\mu_k^* \mathscr{O}_{X_k}(k') \otimes \mathscr{O}_{\widehat{X}_k}(-E)$.

Deformation of Fermat type hypersurfaces

Consider an equation of degree $\varepsilon + (r+k)\delta$ the form

$$F = \sum_{i_0+\cdots+i_n=\delta} a_{i_0,\ldots,i_n} z_0^{(r+k)i_0} \cdots z_n^{(r+k)i_n} = \sum_{|I|=\delta} a_I Z^{(r+k)I}.$$

where ε and δ are "small" integer and r is a "large" integer. The point of this equation is that by differentiating it up to k times, the differentials and the initial equation look alike.

Deformation of Fermat type hypersurfaces

$$
\begin{aligned}
F &= \sum_{|I|=\delta} a_I Z^{(r+k)I} = \sum_{|I|=\delta} \alpha_I^0 Z^{rI} = \sum_{|I|=\delta} \alpha_I^0 T^I \\
dF &= \sum_{|I|=\delta} d(a_I Z^{(r+k)I}) = \sum_{|I|=\delta} \alpha_I^1 Z^{rI} = \sum_{|I|=\delta} \alpha_I^1 T^I \\
\vdots &\quad \vdots \\
d^k F &= \sum_{|I|=\delta} d^k(a_I Z^{(r+k)I}) = \sum_{|I|=\delta} \alpha_I^k Z^{rI} = \sum_{|I|=\delta} \alpha_I^k T^I
\end{aligned}
$$

On the left these are more or less equations for X_k, on the right, these are more or less the equations of a universal complete intersection of multidegree $(\delta, \ldots, \delta)$.

Deformation of Fermat type hypersurfaces

Let $\mathscr{Y} \subset \mathrm{Gr}_{k+1}(H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(\delta))) \times \mathbb{P}^n$ be the universal complete interesection of codimension $k+1$ and multidegree $(\delta, \cdots, \delta)$

$$\mathscr{Y} := \left\{(G, z) \in \mathrm{Gr}_{k+1}(H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(\delta))) \times \mathbb{P}^n;\ P(z) = 0 \ \ \forall P \in G\right\}.$$

Let $p_1 : \mathscr{Y} \to \mathrm{Gr}_{k+1}(H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(\delta)))$ denote the fist projection and $p_2 : \mathscr{Y} \to \mathbb{P}^n$ denote the second projection.
For any $m \in \mathbb{N}$ let

$$\mathscr{O}_{\mathscr{Y}}(m, -1) = p_1^* \mathscr{O}_{\mathrm{Gr}}(m) \otimes p_2^* \mathscr{O}_{\mathbb{P}^n}(-1)$$

be the m-th power of the Plücker line bundle on the Grassmanian twisted by the tautological line bundle on $\mathbb{P}^n$

Deformation of Fermat type hypersurfaces

The point is that if one verifies everything, then the above equations will provide a map

$$\Psi : \widehat{H}_k \to \mathscr{Y}$$

satisfying

$$\Psi^*\mathscr{O}_{\mathscr{Y}}(m,-1) = \nu_k^*\mathscr{O}_{H_k}(mk')\otimes\mathscr{O}_{\widehat{H}_k}(-mE)\otimes\pi_k(m(k+1)(\varepsilon+k\delta)-r)$$

and therefore, for $r \gg 1$, the twist will be negative. Therefore the positivity of the tautological bundle on $\widehat{H}_k$ can be deduced from the positivity of $p_1^*\mathscr{O}_{\mathrm{Gr}}(1)$. But if $k \geqslant n-1$, then the morphism p_2 is generically finite, and therefore $p_1^*\mathscr{O}_{\mathrm{Gr}}(1)$ is big and nef, but moreover, one knows exactly what its augmented base locus is, it is the locus of positive dimensional fibers of p_1.

Deformation of Fermat type hypersurfaces

Using this idea, and solving many technical details, one is able to prove that for suitable constants ε, δ, r and suitable coefficients a_I, the constructed hypersurface satisfies the desired ampleness of

$$\mu_k^* \mathscr{O}_{H_k}(k') \otimes \mathscr{O}_{\widehat{H}_k}(-E),$$

and therefore yields a proof of the Kobayashi conjecture.

Variational method

Let us now conclude by a brief account on the variational method.

Differentiating jet differentials

The main idea of the variational method is to start with one jet differential, and then to differentiate it with meromorphic vector fields. To be more precise.

Theorem

Suppose that we have a variety X with two ample line bundles A, B such that.

1. *There exists $\omega \in H^0(X, E_{k,m}\Omega_X \otimes A^{-1})$*
2. *The tangent space $TJ_kX \otimes p_k^*B$ is globally generate over J_kX^{reg} by $\mathbb{G}_k$ invariant vector fields.*
3. *$A \otimes B^{-m}$ is ample*

Then every entire curve in X is contained in the locus $Z = (\omega = 0) \subset X$.

Idea of Proof

Let $f:\mathbb{C}\to X$ be such that $f(\mathbb{C})\not\subset Z$. Suppose $f(0)\notin Z$. By the fundamental vanishing theorem, $j_kf(0)\in(\omega=0)$, where ω is seen as a function $J_kX\to A^{-1}$. For any $\mathbb{G}_k$ invariant vector field in $TJ_kX\otimes p_k^*B$, one can produce a new invariant jet differential

$$L_V\omega\in H^0(X,E_{k,m}\Omega_X\otimes A^{-1}\otimes B).$$

Applying inductively this differential procedure, one can find $p\leqslant m$ invariant vector field $V_1,\dots,V_p$ in $TJ_kX\otimes p_k^*B$, such that

$$L_{V_1}\cdots L_{V_p}\omega\in H^0(X,E_{k,m}\Omega_X\otimes A^{-1}\otimes B^p)$$

doesn't vanish at $j_kf(0)$. Since $p\leqslant m$ and $A\otimes B^{-m}$ is ample, we also have $A^{-1}\otimes B^p$ is antiample. Therefore, the fundamental vanishing theorem gives a contradiction.

Differentiating in families

In general, this strategy doesn't work because one doesn't have the crucial vector fields. But for hypersurfaces of high degree, one can work over the universal family to get a similar argument.
Let

$$\mathscr{H} \subset \mathbb{P}(H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d))) \times \mathbb{P}^n$$

be the universal hypersurfaces of degree d, namely

$$\mathscr{H} = \{(P, z) \; ; \; P(z) = 0\} .$$

Let us denote by $\rho_1 : \mathscr{H} \to \mathbb{P}^{N_d} := \mathbb{P}(H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d)))$ the first projection and $\rho_2 : \mathscr{H} \to \mathbb{P}^n$ the second projection.

Relative jet spaces

The jet spaces of every hypersurface fit into a family, the space of relative jet spaces

$$p_k : J_k^{\mathrm{rel}}\mathscr{H} \to \mathscr{H}$$

We will redo the same argument as before, starting with a universal jet differential equation, and working with so called "slanted" vector fields.

Slanted vector fields

One has the following theorem.

Theorem (Merker)

For $k = n - 1$, the twisted tangent bundle

$$T_{J_k^{\mathrm{rel}}\mathscr{H}} \otimes \rho_1^* \mathscr{O}(n^2 + 2n) \otimes \rho_2^* \mathscr{O}(1)$$

is globally generated over the regular part $J_k^{\mathrm{rel}}\mathscr{H}^{\mathrm{reg}}$ by global $\mathbb{G}_k$ invariant vector fields.

This result is proven by constructing explicitly generating vector fields.

Existence of jet differentials

The second ingredient is the existence of a universal jet differential with a (very) negative twist. A result in this direction is the following

Theorem (Diverio-Merker-Rousseau)

For any $\delta > 0$ small enough, there exists d_n such that for any smooth hypersurface $H \subset \mathbb{P}^n$ of degree $d \geqslant d_N$, for any m large and divisible enough, one has

$$H^0(H, E_{n-1,m}\Omega_H \otimes K_H^{-\delta m}) \neq 0.$$

By general arguments, there exists a universal such differential form over some non-empty open subset $U \subset \mathbb{P}(H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d)))$, i.e. one that lives on $J_k^{\mathrm{rel}}\mathscr{H}$.

Idea of proof

It is based on the algebraic version of the holomorphic Morse inequalities, the relevent special case of which is the following.

Theorem (Holomorphic Morse inequalities)

Let Y be a projective manifold of dimension m. Let F, G be two nef line bundles. If

$$F^m - mF^{m-1} \cdot G > 0$$

then $F \otimes G^{-1}$ is big.

Idea of proof (continued)

This can be used to prove for instance that $\mathcal{O}_{H_k}(1)$ is big if H is a hypersurface of degree large enough. Using for instance

$$F = \mathcal{O}_{H_k}(2 \cdot 3^{k-2}, 2 \cdot 3^{k-3}, \ldots, 6, 2, 1) \otimes \pi_k^* \mathcal{O}_H(2 \cdot 3^{k-1})$$

and

$$G = \pi_k^* \mathcal{O}_H(2 \cdot 3^{k-1}).$$

A (highly non-trivial) computation will show that the holomorphic Morse inequalities imply that

$$\mathcal{O}_{H_k}(2 \cdot 3^{k-2}, 2 \cdot 3^{k-3}, \ldots, 6, 2, 1)$$

is big, which also implies that $\mathcal{O}_{H_k}(1)$ is big.

Algebraic degeneracy and hyperbolicity

With those two ingredients, the differentiation argument with respect to the slanted vector fiels, working out all the details on the exact values needed for the twists, proves the algebraic degeneracy of entire curves in general hypersurfaces.
More recently, Riedl and Yang proved that up to increasing even more the degree, one can deduce from this the hyperbolicity for general hypersurfaces (i.e. the Kobayashi conjecture).
Even more recently Berczi and Kirwan announced a spectacular improvement on the bound, by replacing the Demailly-Semple jet space by another compactification coming from non-reductive GIT theory in order to perform the intersection computation using the holomorphic Morse inequalities.