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Positivity of the cotangent bundle

Talk IV

Lemma (HP '17)

C smooth proj. curve, $V \to C$ vector bundle with $c_1(V) = 0$

Assume $S^l V$ is stable $\forall\, l \in \mathbb{N}$.

Let $Z \subsetneq \mathbb{P}(V)$ be a subvariety. Then

$$\mathcal{O}_{\mathbb{P}(V)}(1)|_Z \text{ is ample}$$

In particular if Z is a curve, then $\xi_V \cdot Z > 0$.

NB: ξ_V is not ample, since $\xi_V^{\mathrm{rk}\, V} = c_1(V) = 0$.

Proof in case Z is a curve, $Z \twoheadrightarrow C$

Ideal sequence

$$0 \to \mathcal{J}_Z \to \mathcal{O}_{\mathbb{P}(V)} \to \mathcal{O}_Z \to 0$$

twist $\mathcal{O}_{\mathbb{P}(V)}(\ell)$

we want to show that this is ample.

$$0 \to \overset{\mathcal{J}_Z(\ell)}{\overset{\shortparallel}{\mathcal{J}_Z \otimes \mathcal{O}_{\mathbb{P}(V)}(\ell)}} \to \mathcal{O}_{\mathbb{P}(V)}(\ell) \to \mathcal{O}_Z(\ell) \to 0$$

By Serre's theorem we know that for $\ell \gg 0$

- $\pi_* \mathcal{J}_Z(\ell)$ and $\pi_* \mathcal{O}_Z(\ell)$ have positive rank.
- $R^1 \pi_* \mathcal{J}_Z(\ell) = 0$

push forward

$$\Rightarrow \quad 0 \to \pi_* \mathcal{J}_Z(\ell) \to S^\ell V \to \pi_* \mathcal{O}_Z(\ell) \to 0.$$

$S^\ell V$ stable $\quad \pi_* \mathcal{J}_Z(\ell) \subsetneq S^\ell V$

$$\mu(\pi_* \mathcal{J}_Z(\ell)) < \mu(S^\ell V) = 0$$

$$\Rightarrow \mu(\pi_* \mathcal{O}_Z(\ell)) > 0$$

$$\Rightarrow \deg(\pi_* \mathcal{O}_Z(\ell)) > 0$$

Moreover if

$$\begin{tikzcd} \pi_* \mathcal{O}_Z(\ell) \arrow[r] & Q \arrow[r] & 0 \\ S^\ell V \arrow[u] \arrow[ur] & & \end{tikzcd}$$

then $\deg Q > 0$ by stability.

Hartshorne

$\Rightarrow$ $\pi_* \mathcal{O}_Z(\ell)$ is ample.

Since $\underbrace{\pi|_Z^{*}\, \pi_{*}\, \mathcal{O}_Z(1)}_{\text{ample}} \twoheadrightarrow \mathcal{O}_Z(1) \to 0$

$\Rightarrow$ ↗ also ample! $\boxed{}$

Now let's focus on Hyperkähler manifolds

By theorem from talk III we know that Ω_X is not psef

If $H \to X$ ample then by Serre $\exists\, m \in \mathbb{N}$ s.t.

$$\Omega_X \otimes H^{\otimes m} \text{ ample, in particular psef.}$$

Question: can we determine the minimal m_0 s.t.

$$\Omega_X \otimes H^{\otimes m_0} \text{ is psef?}$$

m_0 is called the pseudoeffective threshold of (X, H).

Remark: in general m_0 is not an integer, might be a positive, real number.

Thm (Gounelas, Ottem 2018)

X a projective K3 surface with $\rho(X)=1$

$H \rightsquigarrow X$ ample generator of Picard group, $d := H^2$

Assume $\frac{d}{2}$ is a square and the Pell equation

$$x^2 - 2d\,y^2 = 5$$ has no integer solution.

Then the pseff threshold of (X, H) is $\frac{2}{\sqrt{\frac{d}{2}}}$

Two surprises:

- $\frac{2}{\sqrt{\frac{d}{2}}} \in \mathbb{Q}$
- $\left(\frac{2}{\sqrt{\frac{d}{2}}} H\right)^2 = \frac{4}{\frac{d}{2}} \cdot d = 8$

Thm 1 (Anella, H, 2019)

X projective K3 surface, $A \rightsquigarrow X$ ample $\mathbb{R}$-divisor class

s.t. $A^2 \geq 8$

Then $\Omega_X \otimes A$ is pseff.

Thm 2 (AH, 2019)

X Hyperkähler manifold, $q_X(.)$ the Beauville quadratic form. Then there exists a constant $c_X > 0$ that depends only on the deformation family of X s.t.

if $A \to X$ is ample $\mathbb{R}$-divisor class s.t. $q_X(A) \geq c_X$

then $\Omega_X \otimes A$ is pseff.

Remark: if $X \sim S^{[n]}$ Hilbert scheme of points on K3 surface,

then c_X is computable (in principle)

$n=1 \quad c_X = 8$

$n=2, n=3 \quad c_X$ is not a rational number.

Sketch of proof of Thm 1:

X proj K3 surface, $A \to X$ ample s.t. $A^2 \geq 8$

For K3 surfaces we know that $\exists$

$$\begin{tikzcd} \mathcal{X} \arrow[d, "\text{smooth proper holom. map.}"] \\ \Delta \end{tikzcd}$$

s.t. $\mathcal{X}_0 = X$

for $t \in \Delta$ very general point $\mathcal{X}_t$ is a K3 surface with $NS(\mathcal{X}_t) = 0 \Rightarrow \mathcal{X}_t$ does not contain any curves.

NB: X_t is not projective, only compact Kähler

Up to making Δ smaller, we can find $\omega \to \mathcal{X}$ diff. form of degree 2 s.t. $c_1(\omega|_{X_0}) = A$

and $\omega|_{X_t}$ is a Kähler form s.t. $\omega_{X_t}^2 \geq 8$.

$$\begin{tikzcd}
\xi_{\mathcal{X}} \text{ tautol. class} \arrow[d] & & \omega|_{X_t} \arrow[d, phantom, "\parallel"] \\
\mathbb{P}(\Omega_{\mathcal{X}/\Delta}) \arrow[r, "\pi"] & \mathcal{X} \arrow[d] & \omega_t \\
& \Delta &
\end{tikzcd}$$

Look at $\xi_{\mathcal{X}} + \pi^*\omega$: if we restrict over X_t, the

restriction is $\underbrace{\xi_{X_t}}_{\text{tautological class on } \mathbb{P}(\Omega_{X_t})} + \pi^* \underbrace{\omega_t}_{\text{Kähler class}}$

Thus if we prove that for $t \in \Delta$ very general

$$\xi_t + \pi^*\omega_t \text{ is pseff,}$$

then $\xi_{X_0} + \pi^*\omega_0 = \xi_{X} + \pi^* A$ is pseff.

Lemma X a K3 surface s.t. $NS(X) = 0$.

$\omega \to X$ a Kähler class. Then

$$\xi_X + \pi^*\omega \text{ is nef} \iff \omega^2 \geq 8.$$

Special case of Demailly-Păun criterion:

X cpt Kähler, $V \to X$ vector bundle

$\omega \to X$ Kähler class, $\xi_V \to \mathbb{P}(V)$ tautological class

$$\underbrace{\xi_V + \pi^*\omega \text{ is nef}}_{\text{analytic defn of nefness}} \iff \forall Z \subset \mathbb{P}(V) \text{ subvariety } \underbrace{\text{one has } (\xi_V + \lambda\pi^*\omega)^{\dim Z}\cdot Z \geq 0.}_{\text{algebraic defn of nefness}}$$

$\forall \lambda \geq 1$

Proof of lemma: we want to use DP criterion.

Fix $Z \subset \mathbb{P}(\Omega_X)$ subvariety.

1st case: $\underline{\dim Z} = 3 \iff Z = \mathbb{P}(\Omega_X)$

$$\underset{<0}{(\xi_X + \pi^*\omega)^3} = \xi_X^3 + 3\xi_X^2\pi^*\omega + \underbrace{3\,\xi_X\underbrace{\pi^*\omega^2}_{\geq 8}}_{\geq 24} + \underset{0}{\underset{\|}{\pi^*\omega^3}}$$

Since $c_1(X)=0$, $c_2(X)=24$

so Grothendieck relation implies $\xi_X^3 = -24$, $\xi_X^2\,\pi^*\omega = 0$

$$\geq -24 + 3 \cdot 8 = 0.$$

$$\begin{tikzcd} Z \arrow[r] \arrow[d, phantom, "\cap"] & pt \arrow[d, phantom, "\cap"] \\ \pi : \mathbb{P}(\Omega_X) \arrow[r] & X \end{tikzcd}$$

2nd case: dim $\underline{Z} = 1$

Since X contains no curves, we have $\pi(Z) = \text{point}$

$\Rightarrow Z = \text{fibre of } \pi \Rightarrow (\xi_X + \pi^* \omega) . Z = 1 > 0.$

3rd case: dim $\underline{Z} = 2$

Since X contains no curves, we have $\pi(Z) \subset X$.

so $Z \xrightarrow{\pi|_Z} X$ is (generically) finite.

Since X contains no curves, the branch divisor is empty.

$\Rightarrow \pi|_Z$ is étale.

but $\pi_1(X) = \{1\}$, so $Z \longrightarrow X$ is birational,

i.e. has degree one.

$Z \subset \mathbb{P}(\Omega_X)$ is a (rational) section of π

$$\begin{tikzcd} Z \arrow[r, phantom, "\subset"] & \mathbb{P}(\Omega_X) \arrow[d, "\pi"] \\ & X \arrow[ul, dashed] \end{tikzcd}$$

$$\Rightarrow [Z] = \zeta_X + \pi^* L \quad \text{with } L \in \underset{\substack{\| \\ 0}}{NS(X)}.$$ assumption

$$\Rightarrow [Z] = \zeta_X.$$

i.e. $\underset{\substack{\| \\ H^0(X, \Omega_X)}}{H^0(\mathbb{P}(\Omega_X), \mathcal{O}_{\mathbb{P}(\Omega_X)}(1))} \neq 0$ does not exist on K3 surface.

$\Rightarrow Z$ does not exist $\blacksquare$

I. X projective K3 surface with $\rho(X) = 1$

$A \to X$ ample $\mathbb{R}$-divisor s.t. $A^2 = 8$.

Assume $\Omega_X \otimes A^\lambda$ psef for some $\lambda < 1$.

(so psef threshold is smaller than the universal bound).

what can we say about the geometry of (X, A)?

Ex 1 (Tihomirov, Welters ~1980)

$X \subset \mathbb{P}^3$ very general quartic,

so X K3 surface with $\rho(X) = 1$

H hyperplane class $H^2 = 4$

Then $\exists\, U \subset \mathbb{P}(\Omega_X)$ the surface of bitangents and $[\alpha] = 6\,\xi_X + 8\,\pi^* H$.

Set $A := (\sqrt{2}\,H)$ so $A^2 = 8$, then U shows that

$$\Omega_X \otimes A^{\frac{4}{3\sqrt{2}}} \text{ is pseff.} \qquad \frac{4}{3\sqrt{2}} < 1.$$

Ex 2 (Anella, H 2022)

$X \xrightarrow{f} \mathbb{P}^2$ double cover branched along a very general sextic curve $B \subset \mathbb{P}^2$.

$\rightsquigarrow X$ K3 surface, $\rho(X) = 1$, $L = f^* H$
ample generator of Pic X
$L^2 = 2$

Let $C_t = f^{-1}(l_t)$ where $l_t \subset \mathbb{P}^2$ that is tangent to the branch curve B.

singular curve in X.

We show that $\exists$ canonical lifting

$$\begin{tikzcd} & \mathbb{P}(\Omega_X) \arrow[d] \\ C_t \arrow[ur, bend left, "s_t"] \arrow[r, phantom, "\subset"] & X \end{tikzcd}$$

and $D = \cup\, s_t(C_t)$ is a surface

$$[D] = 30\,\xi_X + 54\,\pi^* L$$

Set $A = 2L$ then A ample $A^2 = 8$ and D shows that $\Omega_X \otimes A^{99}$ is psef.

III. X a Calabi-Yau manifold, so $K_X \simeq \mathcal{O}_X$

$\pi_1(X) = 1$

$h^0(X, \Omega_X^p) = 0$

if $p \neq 0$, $p \neq \dim X$

By HP, Ω_X is not psef.

Q: Is there an analogue of Thm 2, is there a $c_X > 0$ that depends only on topological invariants s.t.

if $A \to X$ ample with $A^{\dim X} \geq c_X$ one has $\Omega_X \otimes A$ psef?

Bad news: C-Y manifolds never have non-projective deformations?

X cpt Kähler surface

$C \subset X$ compact curve

$[C]\,[\omega] > 0.$

/

Kähler class

$\Rightarrow [C] \in NS(X)$

$\neq$

0