SCMS summer school 2022 (8.8 ~ 8.19)

Linear systems on K3 & HK : Plan

Lecture 1 . 8.12 Basic properties of linear systems

Lecture 2 . 8.15 Linear systems on K3

Lecture 3 . 8.18 Linear systems on HK, I

Lecture 4 . 8.19 Linear systems on HK, II.

X: (projective) smooth variety / $\mathbb{C}$.

{prime divisor} = {closed subvariety of codim 1 $\subseteq X$}

{(Weil) divisor} = {free abelian group gen by prime divisors}

$$D = \sum_{i=1}^{r} a_i P_i \qquad a_i \in \mathbb{Z}.$$

principal divisor

$f \in K(X)^*$: function field = {rational functions on X}

P: prime divisor of $X \rightsquigarrow \mathcal{O}_{X,P}$: regular ring of dim 1 $\Rightarrow$ DVR

$\rightsquigarrow v_P : K(X) \longrightarrow \mathbb{Z}$ valuation.

$\rightsquigarrow \mathrm{div}(f) := \sum_P v_P(f) \cdot P$ principal divisor defined by f.

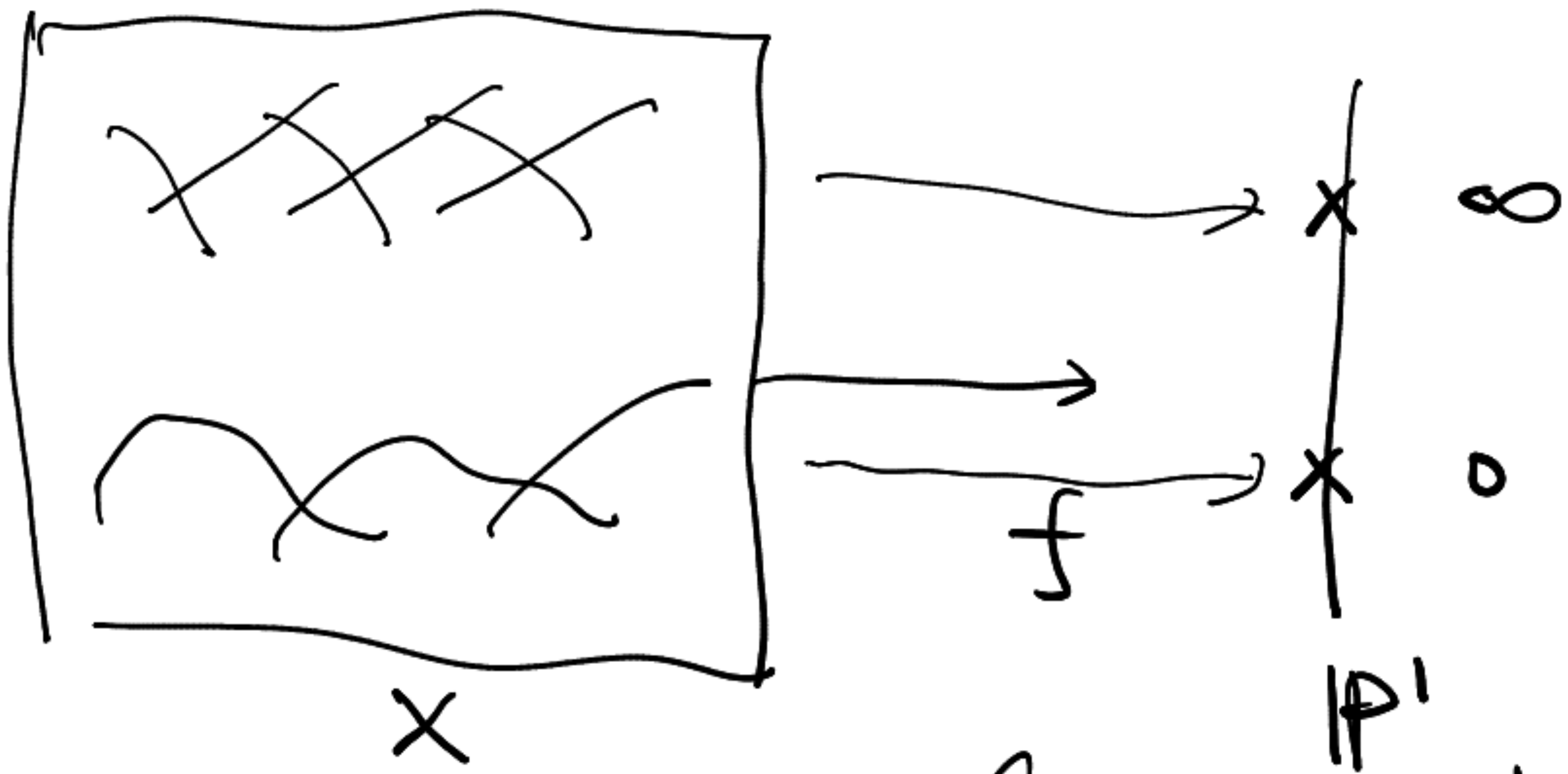
$v_P(f) > 0 \iff P$ is zero of f.

$v_P(f) < 0 \iff P$ is pole of f.

geometrically, $f \in K(X)^* \rightsquigarrow X \overset{f}{\dashrightarrow} \mathbb{A}^1$ rational map.

$\rightsquigarrow X \xrightarrow{f} \mathbb{P}^1$ regular map (in gen, not regular)

$$\boxed{\mathrm{div}(f) := f^{-1}(0) - f^{-1}(\infty)}$$



e.g.: $X = \mathbb{P}^2_{x,y,z}$. $K(X) = \left\{ \dfrac{g(x,y,z)}{h(x,y,z)} \;\middle|\; g, h \text{ homog. } \deg g = \deg h \right.$

$\underset{\shortparallel}{f} \in K(X)^*$, $\operatorname{div}(f) = Z(g=0) - Z(h=0)$

g/h

linearly equivalence: $D \sim D' \iff D - D' = \operatorname{div}(f)$

effective divisor: $D = \sum a_i D_i$, $a_i \geq 0$

linear system: fix D: divisor on X

$|D| := \{ D' \geq 0 \mid D' \sim D \}$ set of divisors

$(\rightsquigarrow D' = D + \operatorname{div}(\underline{\underline{f}}))$

$\downarrow$

$H^0(X, D) := \{ f \in K(X)^* \mid D + \operatorname{div}(f) \geq 0 \} \cup \{0\} \subseteq K(X)$

$\mathbb{C}$-subspace of rational functions.

$\boxed{\text{Exercise}}$
- Show that $H^0(X, D)$ is a subspace
- show $\mathcal{O}_X(D)$ is a line bundle

$$\begin{array}{ccc} \boxed{(H^0(X,D) \setminus 0)/\mathbb{C}^*} & \xrightarrow{\cong} & |D| \quad \leftarrow \text{projective space structure.} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ f & \longmapsto & D + \operatorname{div}(f) \end{array}$$

divisorial sheaf: $\mathcal{O}_X(D)$: line bundle = locally free of rank 1.

$\forall\, U \subseteq X$ open, $\mathcal{O}_X(D)(U) := H^0(U, D)$

$\;\;\neq \emptyset$

$:= \{ f \in K(X)^* \mid D|_U + \operatorname{div}(f)|_U \geq 0 \}$

$\boxed{\text{Ex}}$ $\{\text{Weil divisor}\}/(\sim) \xrightarrow{1:1} \{\text{line bundles}\}/\cong$

$D \longmapsto \mathcal{O}_X(D)$

linear equivalence

isom of $\mathcal{O}_X$-modules.

Interesting: $\begin{cases} \text{geom of a single (effective) divisor} \\ \text{geom of linear system} \\ \text{geom of line bundles} \end{cases}$

Example (conic curves) $X = \mathbb{P}^2_{x,y,z}$ $\quad H = (x=0)$

- $|2H| = \{ D \geq 0 \mid 2H \sim D \}$ $\quad D - 2H = \operatorname{div}\left(\frac{g}{x^2}\right)$

$= \{ D = (g=0) \mid \deg g = 2 \}$

set of conic curves on $\mathbb{P}^2$.

- $H^0(X, 2H) = \left\{ \frac{g(x,y,z)}{x^2} \,\middle|\, \deg g = 2 \right\} = \operatorname{Span}_{\mathbb{C}} \left\{ \frac{x^2}{x^2}, \frac{y^2}{x^2}, \frac{z^2}{x^2}, \frac{xy}{x^2}, \frac{yz}{x^2}, \frac{xz}{x^2} \right\}$

- $|2H| \cong \mathbb{P}^{6-1} = \mathbb{P}^5$.

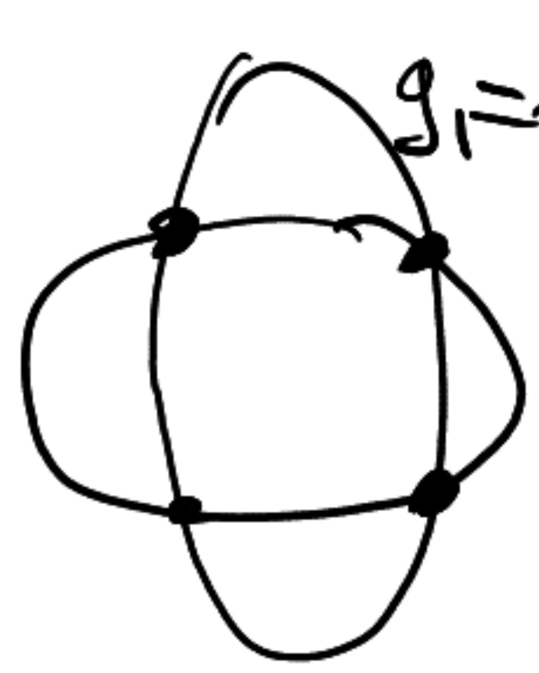
sub linear system. $V \subseteq H^0(X, D)$: sub-$\mathbb{C}$-space

$\rightsquigarrow$

$$\mathbb{P}(V) \cong |\Lambda| \subseteq |D| \cong \mathbb{P}(H^0(X, D))$$

$$|\Lambda| = \{ \operatorname{div}(f) + D \mid f \in V \}.$$

e.g $\mathbb{P}^2$



$V := \operatorname{Span}\{g_1, g_2\}$

$|\Lambda| = \{ (s g_1 + t g_2 = 0) \mid s, t \in \mathbb{C} \}$.

1-dim linear system of conic curves.

$\begin{cases} \text{4 pts} \rightsquigarrow \text{1-dim linear system} \\ \text{5 pts} \rightsquigarrow \text{1 conic curve} \end{cases}$

linear system & rational map $\cong \mathbb{P}^{\dim V-1}$

$X,\ V \subseteq H^0(X, D) \rightsquigarrow |\Lambda| \subseteq |D| \quad \dim V \geq 2$

$\Longrightarrow X \overset{\Phi_{|\Lambda|}}{\dashrightarrow} \mathbb{P}^{\dim V-1} \qquad V = \mathrm{Span}\{g_0, \dots, g_d\}$

$x \longmapsto [g_0(x) : g_1(x) : \cdots : g_d(x)]$ rational map.

$\Phi_{|\Lambda|}$: well-defined on $x \Leftarrow g_0(x), \dots, g_d(x)$ not all 0.

$\mathrm{Bs}|\Lambda| := \bigcap_{D' \in |\Lambda|} \mathrm{Supp}(D')$ base locus of $|\Lambda|$

$\Phi_{|\Lambda|} : X \setminus \mathrm{Bs}|\Lambda| \longrightarrow \mathbb{P}^{\dim V-1}$: regular map.

Def : $\mathrm{Bs}|\Lambda| = \emptyset \overset{\text{def}}{\iff} |\Lambda|$: free.

Example $\mathbb{P}^2$, $2H$. $H^0(X, 2H) = \mathrm{Span}\left\{\frac{x^2}{X^2}, \frac{y^2}{X^2}, \frac{z^2}{X^2}, \frac{xy}{X^2}, \frac{yz}{X^2}, \frac{xz}{X^2}\right\}$

$X \xrightarrow{\Phi_{|2H|}} \mathbb{P}^5$ regular map.

$[x:y:z] \longmapsto \left[\frac{x^2}{\cancel{X^2}} : \frac{y^2}{\cancel{X^2}} : \frac{z^2}{\cancel{X^2}} : \frac{xy}{\cancel{X^2}} : \frac{yz}{\cancel{X^2}} : \frac{xz}{\cancel{X^2}}\right]$

$= [x^2 : y^2 : z^2 : xy : yz : xz]$.

$V = \mathrm{Span}\left\{\frac{g_1}{X^2}, \frac{g_2}{X^2}\right\}$ -

$X \overset{\Phi_{|\Lambda|}}{\dashrightarrow} \mathbb{P}^1$ well-defined except on

$x \dashrightarrow [g_1(x) : g_2(x)]$ $\qquad \underline{\{g_1 = g_2 = 0\}}$: 4 pts.

$\mathrm{Bs}|\Lambda| = (g_1 = g_2 = 0)$

Tools to study freeness of divisors :

① Riemann-Roch formula

② Vanishing theorems.

③ induction on dimension.

Fixed part of $|\Lambda|$

$:= F = \bigcap_{D' \in \Lambda} D'$

$|\Lambda| = |M| + F$

Example: (Curves) X : sm. proj curve genus g.

Riemann-Roch: $\chi(X, D) = \deg D + 1 - g$

$$\underset{\displaystyle h^0(X,D) - \underbrace{h^1(X,D)}_{\displaystyle \overset{\shortparallel}{h^0(X, K_X - D)}}}{\shortparallel}$$

$\deg D > 2g-2 = \deg K_X \Rightarrow h^0(X, K_X - D) = 0$

$\Rightarrow h^0(X, D) = \deg D + 1 - g$.

$x \in X$ point $x \notin Bs|D| \Leftrightarrow |D - x| \subsetneq |D|$

$\Leftrightarrow h^0(X, D-x) < h^0(X, D)$

Thm $\deg D \geq 2g \Rightarrow |D|$: free.

In general,

Conjecture: (Fujita) $\begin{cases} X : \text{sm projective variety } \dim = n. \\ A : \text{ample divisor} \end{cases}$

$\Rightarrow K_X + mA$: free $\forall m \geq n+1$.

e.g.: $X \cong \mathbb{P}^n$. $\begin{cases} K_X \sim -(n+1)H \\ H = A. \end{cases}$

$\boxed{\text{Ex}}$ (use Castelnuovo-Mumford regularity).

Baby version A : free $\Rightarrow K_X + mA$: free $\forall m \geq n+1$.

A "proof": pick $Y \in |A|$ general. $\xrightarrow[\text{Bertini}]{}$ Y : sm. proj var $\dim = n-1$

$0 \to \mathcal{O}_X(K_X + (m-1)A) \to \mathcal{O}_X(K_X + mA) \to \mathcal{O}_Y(K_Y + (m-1)A|_Y) \to 0$

$H^1(X, K_X + (m-1)A) = 0$ (Kodaira vanishing)

$\Rightarrow H^0(X, K_X + mA) \twoheadrightarrow H^0(Y, \underbrace{K_Y + (m-1)A|_Y}_{})$.

$\Rightarrow Bs|K_X + mA| \cap Y = \phi$. free as $m-1 \geq \dim Y + 1$

1. Day 1

We always work over base field $\mathbb{C}$.

Exercise 1.1. Let X be a smooth projective variety and let D be a Weil divisor.

(1) Show that
$$H^0(X, D) := \{f \in K(X)^* \mid \operatorname{div}(f) + D \geq 0\} \cup \{0\}$$
is a $\mathbb{C}$-subspace of $K(X)$.

(2) Show that $\mathcal{O}_X(D)$ defined by
$$\mathcal{O}_X(D)(U) := H^0(U, D)$$
is a line bundle (i.e., locally free sheaf of rank 1).

(3) Show that $D \mapsto \mathcal{O}_X(D)$ gives a 1-1 correspondence between
$$\{\text{Weil divisors}\}/\sim \longleftrightarrow \{\text{line bundles}\}/\simeq.$$

Exercise 1.2. Let X be a smooth projective variety and let $\Lambda \subset |D|$ be a sub-linear system. Show that there is a unique effective divisor F satisfying the following conditions:

(1) for any $D' \in \Lambda$, $D - F \geq 0$
(2) for the induced sub-linear system $M := \Lambda - F$, $\mathrm{Bs}M$ contains no divisor.

Exercise 1.3. Let X be a smooth projective variety and let $\Lambda \subset |D|$ be a sub-linear system. Suppose $\Lambda = M + F$ where M is the movable part and F is the fixed part. Show that $\Phi_\lambda = \Phi_M$.

Exercise 1.4. Let X be a smooth projective variety of dimension n and let A be a free ample divisor. Show that $K_X + mA$ is free for $m \geq n+1$. (Hint: use Castelnuovo–Mumford regularity)