Introduction to hyperbolicity
Lecture 1 : First definitions and properties

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Goal of these lectures

- Basic introduction hyperbolic manifolds
- Some classical properties of hyperbolic manifolds
- Algebraic aspects of the theory
- Hyperbolicity of hypersurfaces

Plan

1. Geometry of the unit disc
   - Poincaré metric
   - Ahlfors-Schwarz
   - The punctured disc
2. Kobayashi hyperbolicity
   - Kobayashi pseudo-distance
   - Kobayashi hyperbolicity
   - Infinitesimal pseudo-metric
3. Montel's theorem
   - Hyperbolically embedded manifolds
   - Classical Montel
   - Arzela-Ascoli
   - Montel and its converse

The Poincaré metric

The starting point is the unit disc

$$\Delta = \{z \in \mathbb{C}\ ;\ |z| < 1\},$$

endowed with the Poincaré metric

$$ds^2 = \frac{dzd\bar{z}}{(1-|z|^2)^2}.$$

That is to say

$$\left\| \frac{\partial}{\partial z} \right\|_{P,z} = \frac{1}{1-|z|^2}.$$

The Poincaré metric distance

Integrating this metric, one defines a distance d_P defined as follows. For any $z_1, z_2 \in \Delta$,

$$d_P(z_1, z_2) = \inf_{\gamma} \int_{\gamma} \gamma^* ds^2 = \inf_{\gamma} \int_0^1 \frac{|\gamma'(t)|}{(1 - |\gamma(t)|^2)} dt$$

where γ ranges over the set of all curves $\gamma : [0,1] \to \Delta$ such that $\gamma(0) = z_1$ and $\gamma(1) = z_2$.

Schwarz-Pick lemma

A key property is the following.

Lemma (Schwarz-Pick lemma)

If $f : \Delta \to \Delta$ is holomorphic, then

$$f^* ds^2 \leqslant ds^2.$$

This precisely means that for any $z \in \Delta$

$$\frac{|f'(z)|^2}{(1 - |f(z)|^2)^2} \leqslant \frac{1}{(1 - |z|^2)^2}.$$

Distance decreasing property

Integrating the previous proposition

Proposition (Distance decreasing)

If $f : \Delta \to \Delta$ is holomorphic, then for any $z_1, z_2 \in \Delta$,

$$d_P(f(z_1), f(z_2)) \leqslant d_P(z_1, z_2).$$

In particular, the Poincaré distance is invariant under automorphisms of Δ.

This can be used to prove that for any $z_1, z_2 \in \Delta$,

$$d_P(z_1, z_2) = \frac{1}{2} \log \left(\frac{|1 - z_1\overline{z_2}| + |z_1 - z_2|}{|1 - z_1\overline{z_2}| - |z_1 - z_2|} \right).$$

Pseudo-metric

Some terminology : Let X be a Riemann surface, denote by T_X its tangent space. A *pseudo-metric* on X is a smooth map $h : T_X \times_X T_X \to \mathbb{C}$ such that for any $x \in X$, the restricted map $h_x : T_{X,x} \times T_{X,x} \to \mathbb{C}$ is a hermitian form on $T_{X,x}$ (It could be vanishing at some points). Set $\|\cdot\|_h : T_X \to \mathbb{R}^+$ the associated length function,

$$\|\xi\|_h = \sqrt{h(\xi,\xi)}$$

for all $\xi \in T_X$.

Let ω_h be the associated $(1,1)$-form on X,

Ricci/Gaussian curvature

Locally, if (z) is a coordinate on X, set $\lambda(z) = \left\| \frac{\partial}{\partial z} \right\|_{h,z}^2$.

Then, on this coordinate chart, $\omega = \frac{i}{2} \lambda dz \wedge d\bar{z}$.

The *Ricci curvature* is:

$$\mathrm{Ric}\,\omega = -2\pi dd^c \log \lambda = -i \frac{\partial^2 \log \lambda}{\partial z \partial \bar{z}} dz \wedge d\bar{z}.$$

This is a global $(1,1)$-form, defined over the open set $(\lambda \neq 0)$.

The *Gaussian curvature* is the function defined by

$$\mathrm{Ric}\,\omega = K\omega, \quad \text{i.e.} \quad K = -\frac{2}{\lambda} \frac{\partial^2 \log \lambda}{\partial z \partial \bar{z}}.$$

With this normalization, the Gaussian curvature K of the Poincaré metric is -4.

Alhfors-Schwarz lemma

The notion of curvature allow us to reinterpret the Schwarz Lemma.

Theorem (Ahlfors-Schwarz Lemma)

Let h be a pseudo-metric on Δ. Let K denotes its Gaussian curvature. If $K \leqslant -4$ whenever defined (or equivalently, if $-\operatorname{Ric}\omega \geqslant 4\omega$ whenever defined), then

$$h \leqslant h_P.$$

This implies Schwarz-Pick: If $f : \Delta \to \Delta$ is holomoprhic, set $h := f^* h_p$, it has curvature -4 outside the ramification points of f.

Proof of Ahlfors-Schwarz

Fix $0 < r < 1$, let $\Delta_r = \{|z| < r\}$ be endowed with the metric

$$ds_r^2 = \lambda_r dzd\bar{z} = \frac{r^2}{(r^2 - |z|^2)^2} dzd\bar{z}.$$

It has also Gaussian curvature -4.
Write $h = \lambda dzd\bar{z}$. On Δ_r, consider

$$u_r(z) = \frac{\lambda(z)}{\lambda_r(z)} = \lambda(z)\frac{(r^2 - |z|^2)^2}{2r^2}.$$

Goal: prove that $u_r(z) \leqslant 1$ for all $z \in \Delta_r$. Then let $r \to 1$.

Proof of Ahlfors-Schwarz

Observe $u_r(z) \to 0$ as $z \to \partial\Delta_r$.
Thus, there exists $z_0 \in \Delta_r$ which is a max for u_r.
If $u_r(z_0) = 0$, then $\lambda \equiv 0$, done.
Suppose $u_r(z_0) > 0$. The z_0 is a local max for $\log u_r$, hence its Hessian matrix is negative definite, therefore

$$\frac{\partial^2 \log(u_r)}{\partial z \partial \bar{z}}(z_0) = \frac{1}{4}\left(\frac{\partial^2 \log(u_r)}{\partial x^2} + \frac{\partial^2 \log(u_r)}{\partial y^2}\right)(z_0) \leqslant 0.$$

Proof of Ahlfors-Schwarz

But one has

$$\begin{aligned}\frac{\partial^2 \log(u_r)}{\partial z \partial \bar{z}} &= \frac{\partial^2 \log \lambda}{\partial z \partial \bar{z}} - \frac{\partial^2 \log \lambda_r}{\partial z \partial \bar{z}} \\ &= -\lambda \left(\frac{-1}{\lambda} \frac{\partial^2 \log \lambda}{\partial z \partial \bar{z}} \right) + \lambda_r \left(\frac{-1}{\lambda_r} \frac{\partial^2 \log \lambda_r}{\partial z \partial \bar{z}} \right) \\ &= -\lambda K_h - 4\lambda_r.\end{aligned}$$

Evaluating at the point z_0,

$$0 \geqslant -\lambda(z_0) K_h(z_0) - 4\lambda_r(z_0) \geqslant 4(\lambda(z_0) - \lambda_r(z_0)).$$

Hence $u_r(z_0) \leqslant 1$, and thus $u_r(z) \leqslant 1$ for all $z \in \Delta_r$.

Slight generalization

By a simple normalization, one obtains

Corollary

Let h be a pseudo-metric on Δ. Let K denotes its Gaussian curvature. If $K \leqslant -4\gamma$ for some $\gamma > 0$ whenever defined (or equivalently, if $-\operatorname{Ric}\omega \geqslant 4\gamma\omega$ whenever defined), then

$$h \leqslant \frac{1}{\gamma} h_P.$$

That is to say, if the curvature is bounded above by a negative constant, then one can compare the given metric with the Poincaré metric.

Poincaré metric on the punctured disc

The Poincaré metric induces a metric on the punctured disc $\Delta^* = \{z \in \mathbb{C}\ ;\ 0 < |z| < 1\}$ via the map

$$\pi : \begin{cases} \Delta & \to & \Delta^* \\ z & \mapsto & e^{\pi \frac{z+1}{z-1}} \end{cases}$$

This map realizes the punctured disc Δ^* as the quotient of Δ by the group action induced by $\varphi : \mathbb{Z} \to \mathrm{Aut}(\Delta)$ mapping k to

$$\varphi_k : z \mapsto \frac{z(1-ik)+ik}{1+ik-ikz}.$$

Proposition

The induced Poincaré metric on the punctured disc is given by

$$h_{\Delta^*} = \frac{2dwd\bar{w}}{|w|^2(\ln|w|^2)^2}.$$

Ahlfors-Schwarz on the punctured disc

Corollary (Ahlfors-Schwarz on the punctured disc)

Let h be a pseudo-metric on Δ^*. Suppose that its curvature K_h is bounded by a negative constant $-4\gamma < 0$. Then

$$h \leqslant \frac{1}{\gamma} h_{\Delta^*}$$

Proof : Let $\pi : \Delta \to \Delta^*$ be the uniformization map. The pull-back $\pi^* h$ on Δ has curvature $K_{\pi^* h}$, also bounded by -4γ.
Ahlfors-Schwarz implies

$$\pi^* h \leqslant \frac{1}{\gamma} h_P = \frac{1}{\gamma} \pi^* h_{\Delta^*}.$$

Both metrics are pull backs of metrics on Δ^*, implying the result

Chain of discs

Let X be a (connected) complex manifold.

Definition (Kobayashi pseudo-distance)

For any $x, y \in X$, a *chain of holomorphic discs* from x to y is a the data γ of a finite sequence of points $x_0, \ldots, x_k \in X$ such that $x = x_0$ and $y = x_k$ and for every $1 \leqslant j \leqslant k$, a holomorphic map $f_j : \Delta \to X$ and two points $z_j, w_j \in \Delta$ such that $f_j(z_j) = x_{j-1}$ and $f_j(w_j) = x_j$. Then *length* of γ is

$$\ell(\gamma) = \sum_{j=1}^{k} d_P(z_j, w_j).$$

Kobayashi pseudo-distance

Definition

For $x, y \in X$, the Kobayashi pseudo-distance between x and y if

$$d_X(x, y) = \inf_{\gamma} \ell(\gamma)$$

where the inf is taken over all chains of holomorphic discs from x to y.

Proposition

The function $d_X : X \times X \to \mathbb{R}_+$ defines a pseudo-distance. Namely

1. $d_X(x, y) = d_X(y, x)$ *for all* $x, y \in X$.
2. $d_X(x, y) \leqslant d_X(x, z) + d_X(z, y)$ *for all* $x, y, z \in X$.

Examples

1. The Kobayashi pseudo-distance of $\mathbb{C}$ is degenerate:

$$d_{\mathbb{C}} \equiv 0.$$

2. Similarly the Kobayashi pseudo-distance of $\mathbb{C}^*$ and the Riemann sphere $S^2 = \mathbb{P}^1_{\mathbb{C}}$ are both degenerate.
3. By Schwarz lemma, the Kobayashi pseudo distance on the disc is equal to the Poincaré distance:

$$d_{\Delta} = d_P.$$

4. More generally, if X is any Riemann surface uniformized by the unit disc, then the Kobayashi distance is the usual hyperbolic metric.

Distance decreasing property

Proposition

Let X and Y be two holomorphic manifolds. Let $f : X \to Y$ be a holomorphic map. For any $x, y \in X$ one has

$$d_Y(f(x), f(y)) \leqslant d_X(x, y).$$

Proof : Any chain of holomorphic discs joining x to y can be composed with f to induce a chain of holomorphic discs joining $f(x)$ to $f(y)$. The result just follows by taking infimums.

Distance decreasing property

The distance decreasing property characterizes the Kobayashi pseudo-distance.

Proposition

Let $\delta : X \times X \to \mathbb{R}_+$ be a pseudo-distance on X such that for any holomorphic maps $f : \Delta \to X$, and for any $z, w \in \Delta$ one has

$$\delta(f(z), f(w)) \leqslant d_\Delta(z, w),$$

then

$$\delta(x, y) \leqslant d_X(x, y), \quad \forall x, y \in X.$$

Proof of the Characterization

Let $x, y \in X$ and consider a chain γ of holomorphic discs given by the data $x = x_0, \ldots, x_k = y$ $f_1, \ldots, f_k : \Delta \to X$ and $(z_i, w_i)_{1 \leqslant i \leqslant k}$ such that $f_i(z_i) = x_{i-1}$ and $f_i(w_i) = x_i$ for all $1 \leqslant i \leqslant k$. Then by definition, the length of γ is given by

$$\ell(\gamma) = \sum_{i=1}^{k} d_\Delta(z_i, w_i).$$

By hypothesis and the triangular inequality, one has

$$\delta(x, y) \leqslant \sum_{i=1}^{k} \delta(x_{i-1}, x_i) = \sum_{i=1}^{k} \delta(f(z_i), f(w_i)) \leqslant \sum_{i=1}^{k} d_\Delta(z_i, w_i) = \ell(\gamma).$$

By passing to the infimum, one obtains

$$\delta(x, y) \leqslant d_X(x, y).$$

More examples

1. The Kobayashi distance on $\Delta \times \mathbb{C}$ is given by

$$d_{\Delta\times\mathbb{C}}\big((z_1,w_1),(z_2,w_2)\big) = d_P(z_1,z_2).$$

2. The Kobayashi distance on $\Delta \times \Delta$ is given by

$$d_{\Delta\times\Delta}\big((z_1,w_1),(z_2,w_2)\big) = \max\left\{d_P(z_1,z_2), d_P(w_1,w_2)\right\}.$$

3. (An example of Eisenman and Taylor). Consider the following open subset of $\mathbb{C}^2$:

$$U = \left\{(z,w)\in\mathbb{C}^2 \; ; \; |z|<1 \text{ and } |zw|<1\right\}\backslash\{(0,w)\; ; \; |w|\geqslant 1\}.$$

For any $(z_1,w_1),(z_2,w_2)\in U$:

$$\begin{aligned} d_U\big((z_1,w_1),(z_2,w_2)\big) &= 0 \text{ if } z_1=0 \text{ and } z_2=0 \\ d_U\big((z_1,w_1),(z_2,w_2)\big) &= d_{\Delta\times\Delta}\big((z_1,z_1w_1),(z_2,z_2w_2)\big) \text{ otherwise.} \end{aligned}$$

Two properties

Proposition

Let X and Y be complex manifolds. For any $x_1, x_2 \in X$ and any $y_1, y_2 \in Y$, one has

$$d_{X\times Y}\big((x_1,y_1),(x_2,y_2)\big) = \max\big\{d_X(x_1,x_2), d_Y(y_1,y_2)\big\}.$$

Proposition

The application

$$d_X : X \times X \to \mathbb{R}_+$$

is a continuous map. Where X is endowed with the Euclidean topology and $\mathbb{R}_+$ with the standard topology.

Kobayashi hyperbolicity

Definition

A complex manifold X is said to be *Kobayashi hyperbolic* if the Kobayashi pseudo-distance d_X is actually a distance. Namely, for all $x, y \in X$

$$d_X(x,y) = 0 \iff x = y.$$

Examples:

1. $\mathbb{C}$, $\mathbb{C}^*$ and $\mathbb{P}^1_{\mathbb{C}}$ are not Kobayashi hyperbolic.
2. Δ is hyperbolic. More generally, any Riemann surface uniformized by the unit disc is Kobayashi hyperbolic (as we shall see).

Functorial properties

Proposition

If X, Y are Kobayashi hyperbolic manifolds, then $X \times Y$ are Kobayashi hyperbolic.

Proposition

Let X be a Kobayashi hyperbolic manifold. Then every submanifold of X hyperbolic.

Proposition

Let X, Y be complex manifolds. Let $f : X \to Y$ be a holomorphic map such that for any $y \in Y$, $f^{-1}(y)$ is finite. If Y is Kobayashi hyperbolic, then X is Kobayashi hyperbolic as well.

Proof

Let $x, y \in X$ such that $x \neq y$. If $f(x) \neq f(y)$: distance decreasing + hyperbolicity Y imply that

$$d_X(x, y) \geqslant d_Y(f(x), f(y)) > 0.$$

Suppose $f(x) = f(y)$. Let U be an open neighborhood of x such that the closure $\overline{U}$ is compact. Suppose moreover that $f^{-1}(f(x)) \cap \overline{U} = \{x\}$ (this is possible by our finiteness assumption). One has $f(\partial U)$ is a compact subset of Y which doesn't contain $f(x)$. In particular

$$d_Y(f(x), f(\partial U)) = \inf_{z \in f(\partial U)} d_Y(f(x), z) > 0.$$

On the other hand, the distance decreasing property implies that

$$d_Y(f(x), f(\partial U)) \leqslant d_X(x, \partial U) \leqslant d_X(x, y).$$

Coverings

Proposition

Let $X, \widetilde{X}$ be complex manifolds and let $f : \widetilde{X} \to X$ be a holomorphic map and suppose moreover that $f : \widetilde{X} \to X$ is also a covering space. Then X is hyperbolic if and only if $\widetilde{X}$ is hyperbolic.

This proves for instance that if X is any Riemann surface uniformized by the unit disc, then X is hyperbolic.

Proof

To show that X hyperbolic implies $\widetilde{X}$ hyperbolic, it is the same proof as the case of finite maps.
Conversely, by the lifting property, one has that for any $x, y \in X$ and for any $\tilde{x} \in \widetilde{X}$ such that $f(\tilde{x}) = x$,

$$d_X(x, y) = \inf_{f(\tilde{y})=y} d_{\widetilde{X}}(\tilde{x}, \tilde{y}).$$

Therefore, if $d_X(x, y) = 0$, there exists a sequence $(\tilde{y}_n)_{n\in\mathbb{N}} \subset \widetilde{X}$, such that $d_{\widetilde{X}}(\tilde{x}, \tilde{y}_n) \to 0$, by continuity this implies $\tilde{y}_n \to \tilde{x}$, hence $x = y$.

Fibrations

Beware that if $f : X \to Y$ is a holomorphic map between two complex manifolds such that Y is hyperbolic and every fibre of f is hyperbolic as well, then X doesn't need to be hyperbolic. For instance:
Let $Y = \Delta$ and let

$$X = (\Delta \times \mathbb{C}) \setminus \big(\{(t, z) \in \Delta \times \mathbb{C} \; ; \; z(t - z) = 0\} \cup \{(0, 1)\} \big).$$

Consider the map $f : X \to \Delta$ given by $f(t, z) = t$. Then, Y is hyperbolic and every fiber of f is biholomorphic to $\mathbb{C} \setminus \{0, 1\}$. However, X is not hyperbolic.

Compact Riemann surfaces revisited

Proposition

Let X be a Riemann surface. If X admits a metric h with Gaussian curvature K bounded by a negative constant $-\gamma < 0$, then X is hyperbolic.

The condition $K < 0$ is not enough. For instance let $\mathbb{C}$ be endowed with the metric $h = (1 + |z|^2)dzd\bar{z}$, then the Gaussian curvature is

$$K = \frac{-2}{(1 + |z|^2)^3} < 0.$$

But $K \to 0$ as $|z| \to +\infty$, and of course, $\mathbb{C}$ is not hyperbolic.

Curves of genus $\geqslant 2$

Proposition

Let X be a compact Riemann surface of genus $g(X) \geqslant 2$, then X admits a metric h with curvature

The proof is very easy. Since $g(X) \geqslant 2$, the canonical bundle K_X is ample. Therefore, one can endow K_X with a smooth metric with positive curvature. Dualizing this metric, we obtain a smooth metric on T_X with negative curvature.

$\mathbb{P}^1 \setminus \{0, 1, \infty\}$

Proposition

On $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, there exists a metric h whose curvature is bounded above by a negative constant.

For instance the metric given by

$$\lambda(z) = \left(\frac{1 + |z|^{\frac{1}{3}}}{|z|^{\frac{5}{3}}} \right) \left(\frac{1 + |z-1|^{\frac{1}{3}}}{|z-1|^{\frac{5}{3}}} \right),$$

will work.

Holomorphic sectional curvature

Let X be a complex manifold. Let h be a metric on T_X, namely a smooth map

$$h : T_X \times T_X \to \mathbb{C}$$

such that for any $x \in X$ the induced map $h_x : T_{X,x} \times T_{X,x} \to \mathbb{C}$ is a hermitian metric.
The *holomorphic sectional curvature of h* is the function (on $\mathbb{P}(\Omega_X)$) defined by

$$\mathrm{HSC}_h(x, [\xi]) = \sup K_{f^*h}(0)$$

for all $x \in X$ and $\xi \in T_{X,x} \setminus \{0\}$, where the supremum is taken over all maps $f : \Delta \to X$ such that $f(0) = x$ and $\xi \in \mathbb{C}f'(0)$.
Observe that if X is a curve, then $\mathrm{HSC}_h = K_h$.

Criterion for hyperbolicity

Theorem

Let X be a complex manifold endowed with a metric h such that $\mathrm{HSC}_h \leqslant -\gamma$ for some $\gamma > 0$. Then X is hyperbolic.

If $\dim X = 1$, one recovers the statement for curves made above.

Lemma

If HSC_h is bounded above by -4, then for any holomorphic $f : \Delta \to X$, one has

$$f^* h \leqslant h_P.$$

Where h_P is the Poincaré metric.

Proof of the lemma: Let $f : \Delta \to X$. One has $K_{f^*h} \leqslant -4$, the Ahlfors Schwarz lemma implies $f^* h \leqslant h_P$.

Proof of the theorem

After normalization, one can suppose $\mathrm{HSC}_h \leqslant -4$. Let δ be the distance induced by h

$$\delta(x, y) = \inf_{\varphi} \int_0^1 \|\varphi'(t)\|_h dt$$

with $\varphi : [0, 1] \to X$ such that $\varphi(0) = x$ and $\varphi(1) = y$. This defines a distance.
Let $f : \Delta \to X$ be a holomorphic map. The lemma tells us that $f^*\| \cdot \|_h \leqslant \| \cdot \|_{h_P}$. Integrating yields, for any $z, w \in \Delta$

$$\delta(f(z), f(w)) \leqslant d_P(z, w).$$

By the distance decreasing characterization of the Kobayashi pseudo-distance, one finds

$$\delta(x, y) \leqslant d_X(x, y)$$

for all $x, y \in X$. Hence d_X is a distance.

Infinitesimal pseudo-metric

Definition

Let X be a complex manifold. The *Kobayashi-Royden infinitesimal pseudo metric* on X is the function $F_X : T_X \to \mathbb{R}_+$ defined by

$$F_X(\xi) = \inf\left\{\|u\|_{h_P,z}\ ;\ f : \Delta \overset{\text{holo}}{\to} X,\ u \in T_\Delta,\ \ df(u) = \xi\right\}$$
$$= \inf\left\{\lambda \in \mathbb{R}_+\ ;\ \exists f : \Delta \overset{\text{holo}}{\to} X,\ \lambda f'(0) = \xi\right\}$$

Holomorphic maps are distance decreasing with respect to it: If $f : X \to Y$ then for any $\xi \in T_X$, one has

$$F_Y(\xi) \leqslant F_X(f_*\xi).$$

Remark : One easily observes that for any $\lambda > F_X(\xi)$ there exists $f : \Delta \to X$ such that $\lambda f'(0) = \xi$.

Examples

1. For any $\xi \in T_\Delta$, $F_\Delta(\xi) = \|\xi\|_{h_P}$.
2. For any $\xi \in T_\mathbb{C}$, $F_\mathbb{C}(\xi) = 0$.

Proposition

Let X, Y be two complex manifolds. Then for any $\xi \in T_X \setminus \{0\}$ and any $\eta \in T_Y \setminus \{0\}$, one has

$$F_{X\times Y}(\xi,\eta) = \max\{F_X(\xi), F_Y(\eta)\}.$$

Here we see (ξ,η) as an element in $T_{X\times Y} \equiv T_X \times T_Y$.

Royden's theorems

F_X is not continuous in general, but

Theorem (Royden)

Let X be a complex manifold. Then F_X is an upper semi-continuous function.

Theorem (Royden)

Let X be a complex manifold. For any $x, y \in X$, one has

$$d_X(x, y) = \inf_{\gamma} \int_0^1 F_X(\gamma'(t))dt$$

where the infimum is taken over all piecewise smooth curves $\gamma : [0, 1] \to X$ joining x to y.

Royden's extension lemma

The proofs of Royden's theorem are rather tedious. Let us just mention that the key ingredient in the proofs is Royden's extension lemma.

Lemma (Royden's extension lemma)

Let X be a complex manifold of dimension n. Let $r > 1$. If $f : \Delta_r \to X$ is a holomorphic map such that $f'(0) \neq 0$, then there exists a holomorphic map $\tilde{f} : \Delta^n \to X$ which is a local biholomorphism at the origin and such that

$$\tilde{f}(z, 0, \dots, 0) = f(z) \ \ \forall z \in \Delta.$$

Moreover, if f is an embedding, then one can chose $\tilde{f}$ to be an embedding as well.

Complements of higher codimensional subsets

Let us conclude by mentioning that codimension at least two subset are irrelevant regarding hyperbolicity

Theorem

Let X be a complex manifold of dimension n. Let $Z \subset A$ be an analytic subset of codimension at least two. Then

$$F_X|_{X \setminus Z} = F_{X \setminus Z}.$$

In particular for any $x, y \in X \setminus Z$,

$$d_X(x,y) = d_{X \setminus Z}(x,y).$$

Idea of proof

By the distance decreasing property, it is clear that

$$F_X|_{X \setminus Z} \leqslant F_{X \setminus Z}.$$

For the other direction, take $x \in X \setminus Z$ and $\xi \in T_{X,x}$. Take a disc $f : \Delta \to X$ close to realizing $F_X(\xi)$. Use Royden's extension lemma to slightly deform f in order to avoid Z and fixing $f'(0)$. The conclusion follows.

Hyperbolically embedded manifolds

Definition

Let X be a compact manifofd. Let $U \subset X$ be an open dense subset of X. Then U is *hyperbolically embedded* in X if for any hermitian metric h on X, there exists $\varepsilon > 0$ such that

$$\varepsilon \| \cdot \|_h |_U \leqslant F_U.$$

Of course, by Royden's theorem, if U is hyperbolically embedded in X then U is Kobayashi hyperbolic. But this notion is stronger, and depends on the embedding. Take X to be a compact hyperbolic manifold and $x \in X$, then $U = X \setminus \{x\}$ is hyperbolically embedded in X, but if $Y = \mathrm{Bl}_x X$ and E is the exceptional divisor, then $U = Y \setminus E = X \setminus \{x\}$ is not hyperbolically embedded in Y.

Classical Montel

Recall the following classical version of Montel's theorem

Theorem (Montel's theorem)

Let $U \subset \mathbb{C}$ be an open subset. Let $\mathscr{F} \subset \mathscr{O}(U)$ be a family of holomorphic functions on U. If each function $f \in \mathscr{F}$ satisfies $f(U) \subset \mathbb{C} \setminus \{0, 1\}$, then $\mathscr{F}$ is relatively compact in $\mathrm{Hol}(U, \mathbb{P}^1)$.

I.e. for every sequence $(f_n)_{n\in\mathbb{N}} \subset \mathscr{F}$, there exists a subsequence $(f_{\varphi(n)})_{n\in\mathbb{N}}$ converging uniformly over compact subsets towards some $f \in \mathrm{Hol}(U, \mathbb{P}^1)$.

Arzela-Ascoli

Recall the

Theorem (Arzela-Ascoli)

Let X and Y be complex manifolds. Suppose that Y is compact and is endowed with a distance function δ_Y. Let $\mathscr{F} \subset \mathrm{Hol}(X, Y)$ be a family of holomorphic maps. If the family $\mathscr{F}$ is equicontinuous on X, then $\mathscr{F}$ is relatively compact in $\mathrm{Hol}(X, Y)$.

This again means that for every sequence $(f_k)_{k\in\mathbb{N}}$ of elements in $\mathscr{F}$ there exists a subsequence $(f_{\varphi(k)})_{k\in\mathbb{N}}$ of $(f_k)_{k\in\mathbb{N}}$ and an element $f \in \mathrm{Hol}(X, Y)$ such that $f_{\varphi(k)}$ converges towards f uniformly on compact subspaces.

Distance decreasing property and equicontinuity

Observe the following lemma

Lemma

Let X and Y be complex manifolds. Suppose that X is endowed with a continuous pseudo-distance function δ_X and that Y is endowed with a distance δ_Y. Let $\mathscr{F} \subset \mathrm{Hol}(X, Y)$. If every element of $\mathscr{F}$ is distance decreasing with respect to δ_X and δ_Y. Then the family $\mathscr{F}$ is equicontinuous on X.

Proof : Let $x \in X$. Let $\varepsilon > 0$. Set $U_\varepsilon := \{z \in X \; ; \; \delta_X(x, z) < \varepsilon\}$. By continuity, U_ε is an open neighborhood of x. By the distance decreasing property, for any $z \in U_\varepsilon$ and for any $f \in \mathscr{F}$, one has

$$\delta_Y(f(z), f(x)) \leq \delta_X(z, y) < \varepsilon.$$

Compact Montel

The lemma and the Arzela-Ascoli theorem imply at once

Theorem (Compact Montel)

Let X be a complex manifold. Let Y be a Kobayashi hyperbolic compact complex. Then $\mathrm{Hol}(X, Y)$ *is compact.*

Proof: Endow X with the (continous) Kobayashi pseudo distance d_X, and Y with the Kobayashi distance d_Y. By the lemma and the distance decreasing property, $\mathrm{Hol}(X, Y)$ is equicontinuous. By Arzela Ascoli, $\mathrm{Hol}(X, Y)$ is relatively compact in $\mathrm{Hol}(X, Y)$, i.e. compact.

General Montel

In general one has:

Theorem (Montel)

Let Y be a compact complex space and let $U \subset Y$ be a dense open subset which is hyperbolically embedded in Y. Then for any complex manifold X, the space $\mathrm{Hol}(X, U)$ is relatively compact in $\mathrm{Hol}(X, Y)$.

Proof of Montel

Let h be a hermitian metric on Y. By definition, there exists $\varepsilon > 0$ such that over U, one has

$$\varepsilon \| \cdot \|_h \leq F_U.$$

Let $d_{\varepsilon,h}$ be the distance on Y obtained by integrating $\varepsilon \| \cdot \|_h$. Let d_X be the Kobayashi pseudo distance on Y. From the distance decreasing property of the Kobayashi pseudo-metric and the fact that d_U is the integrated form of F_U, we see that every element in $f \in \mathrm{Hol}(Y, U)$ is a distance decreasing function $f : (Y, d_Y) \to (X, d_{\varepsilon,h})$. By the lemma, $\mathrm{Hol}(X, U)$ is equicontinuous, and by Arzela-Ascoli theorem, $\mathrm{Hol}(X, U)$ is relatively compact in $\mathrm{Hol}(X, Y)$.

Converse to Montel

One actually has the converse as well.

Theorem

Let Y be a compact complex space and let $U \subset Y$ be a dense open subset. Suppose that for any complex manifold X, the space $\mathrm{Hol}(X, U)$ is relatively compact in $\mathrm{Hol}(X, Y)$. Then U is hyperbolically embedded in Y.

Proof of the converse

Let h be a hermitian metric on X. Assume that U is not hyperbolically imbedded in Y. By definition, for any $n \in \mathbb{N}^*$ there exists $\xi_n \in T_U$ such that $F_U(\xi_n) \leqslant \frac{1}{n}\|\xi_n\|_h$. Therefore, for each n, there exists a holomorphic map $f_n : \Delta \to U$ such that

$$\|f_n'(0)\|_h = \frac{n}{2}.$$

If $\mathrm{Hol}(\Delta, U)$ was relatively compact in $\mathrm{Hol}(\Delta, X)$, then up to extracting a subsequence $(f_{\varphi(n)})_{n\in\mathbb{N}}$ one would obtain a map $f : \Delta \to X$ such that $f_{\varphi(n)}$ converges towards f uniformly on compact subsets. In particular, this would imply that

$$\lim_{n\to+\infty} \|f_{\varphi(n)}'(0)\|_h = \|f'(0)\|_h.$$

But this is impossible since we have

$$\lim_{n\to+\infty} \|f_{\varphi(n)}'(0)\|_h = +\infty.$$

Examples of hyperbolically embedded varieties

The first example is $\mathbb{P}^1 \setminus \{0, 1, \infty\} \subset \mathbb{P}^1$. More interestingly, take $L_1, \ldots, L_6 \subset \mathbb{P}^1$ be the lines as below. Then $U = \mathbb{P}^2 \setminus L_1 \cup \cdots \cup L_6$ is hyperbolically embedded in $\mathbb{P}^2$.