Introduction to hyperbolicity
Lecture 2 : Entire curves and Picard's theorems

Damian Brotbek

August 15, 2022
Shanghai Center for Mathematical Sciences

Plan

Picard theorems

Recall the following two results.

Theorem (Picard Little Theorem)

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire curve. Suppose that $f(\mathbb{C})$ omits at least two points. Then f is constant.

Theorem (Picard Big Theorem)

Let $f : \Delta^* \to \mathbb{C}$ be a holomorphic function. If f is not meromorphic at 0, then $f(\Delta^*)$ omits at most two points.

Hyperbolic implies little Picard

Let us start with a definition

Definition

Let X be a complex manifold, an *entire curve in* X is a non-constant holomorphic map $f : \mathbb{C} \to X$.

The first trivial observation is that

Proposition

If X is Kobayashi hyperbolic, then X doesn't contain any entire curve.

Remark that since we know that $\mathbb{P}^1 \setminus \{0, 1, \infty\} = \mathbb{C} \setminus \{0, 1\}$ is hyperbolic, this observation implies Picard's little theorem.
Proof : f is distance decreasing with respect to $d_{\mathbb{C}} \equiv 0$ and the distance d_X.

Brody hyperbolicity

This justifies the definition

Definition

A complex manifold X is said to be *Brody hyperbolic* if X doesn't contain any entire curve.

We thus have proven that

$$\textit{Kobayashi hyperbolic} \Rightarrow \textit{Brody hyperbolic}.$$

The converse doesn't always holds. A counter example is given by the Eisenmann-Taylor example:

$$U = \left\{(z,w) \in \mathbb{C}^2 \ ; \ |z| < 1 \text{ and } |zw| < 1\right\} \setminus \left\{(0,w) \ ; \ \ |w| \geqslant 1\right\}.$$

U is Brody hyperbolic but not Kobayashi hyperbolic.

Brody's reparametrization lemma

Lemma (Brody's reparametrization lemma)

Let X be a complex manifold endowed with a hermitian metric h. Let $f : \Delta \to X$ be a holomorphic map. For any $0 < r < 1$ there exists $R \geqslant r\|f'(0)\|_h$ and a biholomorphic map $\varphi : \Delta_R \to \Delta_r$ such that

1. $\|(f \circ \varphi)'(0)\|_h = 1$
2. *For all $t \in \Delta_R$,* $\|(f \circ \varphi)'(t)\|_h \leqslant \dfrac{R^2}{(R^2 - |t|^2)}$.

If we denote by $\| \cdot \|_{\Delta_R,h}$ the operator norm induces by the Poincaré metric $h_{\Delta_R} = \frac{R^2 dz d\bar{z}}{(R^2 - |t|^2)^2}$ and the metric h on X, those two properties can be reformulated by

$$\|df_0\|_{\Delta_R,h} = R \quad \text{and} \quad \|df_t\|_{\Delta_R,h} \leqslant R \ \forall\, t \in \Delta_R$$

Proof

Fix $0 < r < 1$. Consider $f_r : \Delta \to X$ defined by

$$f_r(z) = f(rz).$$

Since $\left\|\frac{\partial}{\partial z}\right\|_\Delta \to +\infty$ as $|z| \to 1$, one has $\|df_{r,z}\|_{\Delta,h} \to 0$ as $|z| \to 1$. Therefore the function $z \mapsto \|df_{r,z}\|_{\Delta,h}$ reaches its maximum at some point $z_0 \in \Delta$. Let ψ be a biholomorphism of Δ such that $\psi(0) = z_0$. For every $z \in \Delta$, since ψ preserve the Poincaré metric, one has

$$\|d(f_r \circ \psi)_z\|_{\Delta,h} = \|df_{r,\psi(z)}\|_{\Delta,h}\|\psi_z\|_{\Delta,\Delta} = \|df_{r,\psi(z)}\|_{\Delta,h} \leqslant \|df_{r,z_0}\|_{\Delta,h}.$$

Set

$$R := \|df_{r,z_0}\|_{\Delta,h} = r\|df_{rz_0}\|_{\Delta,h} = r(1-|z_0|^2)\|f'(rz_0)\|_h.$$

Proof

By our choice of z_0, one has $R \geqslant \|df_{r,0}\|_{\Delta,h} = r\|f'(0)\|_h$. Let $g_R : \Delta_R \to \Delta$ be the map defined by $g_R(t) = \frac{t}{R}$. By construction of the Poincaré metric on Δ_R, $\|dg_R\|_{\Delta_R,\Delta} = 1$. Let $\varphi : \Delta_R \to \Delta_r$ be the map defined by

$$\varphi(t) = r\psi(g_R(t)) = r\psi(\frac{t}{R})$$

the map $f \circ \varphi = f_r \circ \psi \circ g_R : \Delta_R \to X$ satisfies

$$\begin{aligned}\|df \circ \varphi_t\|_{\Delta_R,h} &= \|d(f_r \circ \psi \circ g_R)_t\|_{\Delta_R,h} \\ &= \|d(f_r \circ \psi)_{g_R(t)}\|_{\Delta,h}\|d(g_R)_t\|_{\Delta_R,\Delta} \leqslant R\end{aligned}$$

and one has also

$$\|df \circ \varphi_0\|_{\Delta_R,h} = \|df_{r,z_0}\|_{\Delta,h} = R.$$

which proves the lemma.

Brody's convergence result

We now have the very important

Theorem (Brody)

Let X be a compact variety endowed with a hermitian metric h. Let $(f_k)_{k\in\mathbb{N}}$ be a sequence of maps $f_k : \Delta \to X$. Suppose that

$$\lim_{k\to\infty} \|f_k'(0)\|_h = +\infty.$$

Then there exists a non-constant holomorphic map

$$f : \mathbb{C} \to X$$

such that

$$\|f'(0)\|_h = 1 \quad \textit{and} \quad \|f'(z)\|_h \leqslant 1 \ \ \forall z \in \mathbb{C}.$$

Proof

Take a sequence $(r_k)_{k\in\mathbb{N}}$ of elements $r_k \in]0,1[$ such that $r_k \to 1$ as $k \to \infty$. By Brody's reparametrization theorem, there exists a sequence $(R_k)_{k\in\mathbb{N}}$ such that $R_k \geqslant r_k \|f_k'(0)\|_h$ and a sequence of biholomorphisms $\varphi_k : \Delta_{R_k} \to \Delta_{r_k}$ such that

$$\|(f_k \circ \varphi_k)'(0)\|_h = 1$$

and

$$\|(f_k \circ \varphi_k)'(t)\|_h \leqslant \frac{R_k^2}{(R_k^2 - |t|^2)} = R_k \left\| \frac{\partial}{\partial t} \right\|_{\Delta_{R_k},t} \quad \forall\ t \in \Delta_{R_k}.$$

Up to extracting a subsequence, we can suppose that $(R_k)_{k\in\mathbb{N}}$ is an increasing sequence.

Proof (continued)

For any $k \leqslant m$ and for any $t \in R_k$ one has

$$R_m \left\| \frac{\partial}{\partial t} \right\|_{\Delta_{R_m}, t} \leqslant R_k \left\| \frac{\partial}{\partial t} \right\|_{\Delta_{R_k}, t}.$$

Therefore, we deduce that for any $k \in \mathbb{N}$ the family

$$\{ f_m \circ \varphi_m|_{\Delta_k} : \Delta_k \to X \ ; \ m \geqslant k \}$$

is equicontinuous on Δ_k with respect to the Poincaré distance and the distance on X induced by h. Therefore, this family is also equicontinuous on Δ_k with respect to the euclidean metric on Δ_k (and the metric induced by h on X).

Proof (continued)

Since X is compact, we can apply the Arzela-Ascoli theorem in order to obtain a subsequence of $(f_m \circ \varphi_m)_{m \geqslant k}$ converging to a holomorphic map

$$g_k : \Delta_{R_k} \to X$$

such that

$$\|g_k'(0)\|_h = 1 \quad \text{and} \quad \|g_k'(t)\|_h \leqslant 1 \;\; \forall t \in \Delta_{R_k}.$$

(Here we used that $\frac{R_m^2}{(R_m^2 - |t|^2)} \to 1$ as $m \to \infty$.)

Proof (end)

To obtain the actual result, one has to make successive extractions as follows. We first apply the previous argument to $k = 1$ extracting a subsequence $(f_{\varphi_1(n)})_{n\in\mathbb{N}} \subset (f_n)_{n\in\mathbb{N}}$ converging to $g_1 : \Delta_{R_1} \to X$. Then we apply the same argument to $k = 2$, by extracting a sequence $(f_{\varphi_2(\varphi_1(n))})_{n\in\mathbb{N}} \subset (f_{\varphi_1(n)})_{n\in\mathbb{N}}$ converging to a holomorphic map $g_2 : \Delta_{R_2} \to X$, such that $g_2|_{\Delta_{R_1}} = g_1$ and such that

$$\|g_2'(0)\|_h = 1 \quad \text{and} \quad \|g_2'(t)\|_h \leqslant 1 \ \ \forall t \in \Delta_{R_2}.$$

Continuing this process for all k, we obtain the desired map $f : \mathbb{C} \to X$.

Brody curve

This motivates

Definition

Let X be a *compact* complex variety. Let h be a hermitian metric on X. A *Brody curve* is a non-constant holomorphic map $f : \mathbb{C} \to X$ such that there exists $C \in \mathbb{R}_+$ such that

$$\|f'(z)\|_h \leqslant C.$$

Since in this definition, X is compact, the notion of a Brody curve is independent of the choice of metric h.

Brody's criterion

We can now state the criterion.

Theorem (Brody's Criterion)

Let X be a compact complex manifold. Let h be a hermitian metric on X. Then the following are equivalent:

1. *X is Kobayashi hyperbolic.*
2. *X is Brody hyperbolic, i.e. there exists no entire curve in X.*
3. *There exists no Brody curve in X.*
4. *X is hyperbolically embedded in X, i.e. There exists $\varepsilon > 0$ such that*

$$\varepsilon \| \cdot \|_h \leqslant F_X.$$

Proof

One only has to prove $\boxed{3 \Rightarrow 4}$. Arguing by contradiction, suppose that for every $n \in \mathbb{N}^*$, there exists $\xi_n \in T_X$ such that

$$F_X(\xi_n) < \frac{\|\xi_n\|_h}{n}.$$

Therefore, for any $n \in \mathbb{N}^*$, there exists a holomorphic map $f_n : \Delta \to X$ such that $\frac{\|\xi_n\|_h}{n} f_n'(0) = \xi_n$. This implies that

$$\|f_n'(0)\|_h = n.$$

Therefore, we have a sequence of maps such that

$$\lim_{n \to +\infty} \|f_n'(0)\|_h = +\infty.$$

Applying Brody's Theorem we obtain a Brody curve in X.

Winkelmann's example

Brody curves behave rather differently than general entire curves. The following example due to Winkelman illustrates this difference in a rather striking way.

Theorem (Winkelmann)

There exists a compact complex manifold X of dimension 3 containing a smooth surface S such that:

1. *Through every point of X passes an entire curves.*
2. *Every Brody curve of X is contained in S.*

Openness of hyperbolicity

As another application of Brody's theorem, let us mention that hyperbolicity is an open condition (in the euclidean topology).

Theorem

Let B be a complex manifold. Let $\pi : \mathscr{X} \to B$ be a smooth family of compact complex manifold (i.e. $\mathscr{X}$ is a complex manifold he map π is holomorphic and for every $b \in B$, the fibre $X_b = \pi^{-1}(\{b\})$ is a compact complex manifold). If there exists $b_0 \in B$ such that X_{b_0} is Kobayashi hyperbolic, then there exists an open neighborhood U of b_0 in B such that X_b is Kobayashi hyperbolic for any $b \in U$.

Proof

Arguing by contradiction. If this is not the case, then there exists a sequence $(t_n)_{n\in\mathbb{N}}$ of points in B converging to b_0 such that for any $n\in\mathbb{N}$, X_{t_n} is not hyperbolic. By Brody's criterion, this implies that for any $n\in\mathbb{N}$, there exists an entire curve $f_n:\mathbb{C}\to X_{t_n}$, and by Brody's lemma, one can furthermore suppose that for all n

$$\|f_n'(0)\|=1 \quad \text{and} \quad \|f_n'(z)\|_h\leqslant 1 \;\; \forall\, z\in\mathbb{C}.$$

Applying the Arzela-Ascoli theorem, one can extract a sub sequence $(f_{\varphi(n)})$ converging, uniformly on compact subsets, to a holomorphic map $f:\mathbb{C}\to\mathscr{X}$. Since $\|f_n'(0)\|_h=1$ for all n, it follows that $\|f'(0)\|_h=1$, hence f is non-constant. Since $t_n\to b_0$ as $n\to\infty$, it follows that $f(\mathbb{C})\subset X_{b_0}$, and therefore X_{b_0} is not hyperbolic, a contradiction.

Hurwitz Theorem

Recall the following theorem from complex analysis.

Theorem (Hurwitz)

Let $U \subset \mathbb{C}$ be a connected open subset. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of holomorphic maps $f_n : U \to \mathbb{C}$ such that $f_n(z) \neq 0$ for all $z \in U$ and for all $n \in \mathbb{N}$. Suppose that f_n converges to a holomorphic function $f : U \to \mathbb{C}$ uniformly on compact subsets. Then either

$$f(z) = 0 \; \forall z \in U \quad \text{or} \quad f(z) \neq 0 \; \forall z \in U.$$

A corollary of Hurwitz

Proposition

Let X be a complex manifold and let $H \subset X$ be a hypersurface. Set $U = X \setminus H$. Let $(R_k)_{k \in \mathbb{N}}$ be an increasing sequence of real numbers and let $(f_k)_{k \in \mathbb{N}}$ be a sequence of maps such that

1. *For any $k \in \mathbb{N}$, one has $f_k : \Delta_{R_k} \to U$.*
2. *There exists a holomorphic map $f : \mathbb{C} \to X$ such that for any $k_0 \in \mathbb{N}$ the sequence $(f_k|_{\Delta_{R_{k_0}}})_{k \geqslant k_0}$ of maps from $\Delta_{R_{k_0}}$ to U converges uniformly on compact subsets to the map $f|_{\Delta_{R_{k_0}}}$.*

Then, either

$$f(\mathbb{C}) \subset H \quad \text{or} \quad f(\mathbb{C}) \subset U.$$

Proof

Let $z_0 \in \mathbb{C}$ such that $f(z_0) \in H$. Let W be a neighborhood of $f(z)$ isomorphic to a polydisc and let $h : W \to \mathbb{C}$ be a holomorphic function such that $H \cap W = \{w \in W \ ; \ h(w) = 0\}$. Let $V \subset \mathbb{C}$ be an open neighborhood of z such that $\overline{V}$ is compact and that $f(\overline{V}) \subset W$. Take k_0 sufficiently large such that $\overline{V} \subset \Delta_{R_{k_0}}$ and such that $f_k(\overline{V}) \subset W$ for all $k \geq k_0$. Consider then for all $k \geq k_0$, the functions $g_k : V \to \mathbb{C}$ defined by

$$g_k = h \circ f_k|_V.$$

The sequence $(g_k)_{k \geq k_0}$ converges uniformly towards $g = h \circ f|_V$. Moreover g_k never vanishes. Therefore, by Hurwitz theorem, since $g(z_0) = 0$, it follows that $g(z) = 0$ (hence $f(z) \subset H$) for all $z \in V$. By analytic continuation, one obtains that $f(\mathbb{C}) \subset H$.

A theorem of Green and Howard

We can now prove the following version of Brody's theorem.

Theorem (Green, Howard)

Let X be a compact complex manifold. Let $H \subset X$ be a hypersurface. Let $U = X \setminus H$. If

1. *there exists no entire curves in U, and*
2. *there exists no entire curves in H,*

then U is hyperbolically embedded in X.

This shows for instance that $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ is hyperbolically embedded in $\mathbb{P}^1$.

Proof

Let h be a hermitian metric on X. By contradiction, if U is not hyperbolically embedded in X then for any $n \in \mathbb{N}$, there exists $\xi_n \in T_U$ such that

$$F_U(\xi_n) < \frac{1}{n}\|\xi_n\|_h.$$

For any $n \in \mathbb{N}^*$ take a holomorphic map $f_n : \Delta \to X$ such that

$$\frac{\|\xi\|_h}{n} f_n'(0) = \xi_n.$$

This implies that

$$\lim_{n\to\infty} \|f_n'(0)\|_h = +\infty.$$

Proof (end)

As in the proof of Brody's theorem, we construct an increasing sequence of real numbers $(R_k)_{k\in\mathbb{N}^*}$ and a sequence of reparametrizations $(\varphi_k)_{k\in\mathbb{N}}$, where $\varphi_k : \Delta_{R_k} \to \Delta$ is a holomorphic map, such that the maps $f_k \circ \varphi_k : \Delta_{R_k} \to U$ converge uniformly on compact subsets to a holomorphic map $f : \mathbb{C} \to X$, satisfying $\|f'(0)\|_h = 1$ (hence f is non-constant). We can therefore apply the consequence of Hurwitz theorem to conclude that either $f(\mathbb{C}) \subset U$, or $f(\mathbb{C}) \subset H$, therefore either (1) or (2) fail.

A generalization

A similar argument yields the following.

Theorem

Let X be a compact complex manifold. Let $D = \sum_{i=1}^{p} D_i \subset X$ be a divisor. If for all $A, B \subset \{1, \ldots, p\}$ such that $A \cup B = \{1, \ldots, p\}$, the strata

$$\bigcap_{i \in A} D_i \setminus \bigcup_{i \in B} D_i$$

doesn't contain any entire curve, then U is hyperbolically embedded in X.

An application

This application is due to Green. Using the so-called Cartan's Second Main Theorem, on can prove that

Theorem (Bloch)

Let $H_1, \ldots, H_{2n+1} \subset \mathbb{P}^n$ be hyperplanes in general position in $\mathbb{P}^n$, then

$$U := \mathbb{P}^n \setminus (H_1 \cup \cdots \cup H_{2n+1})$$

is Brody hyperbolic.

As a consequence, of the previous result, one deduces that U is hyperbolically embedded in $\mathbb{P}^n$.

Classical Great Picard

The classical great Picard theorem can be rephrased in terms of an extension property.

Theorem

Any holomorphic map $f : \Delta^* \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$ *extends to a map* $\tilde{f} : \Delta \to \mathbb{P}^1$.

We will now see how this is also a hyperbolicity property.

Kwack's Extension result

The following extension result is due to Kwack.

Theorem (Kwack)

Let X be a compact complex manifold. Let $f : \Delta^* \to X$ be a holomorphic map. Suppose that there exists a distance function δ_X on X such that f is distance decreasing with respect to the Poincaré metric on the punctured disc and the metric δ_X on X. Then f extends to a holomorphic map $\tilde{f} : \Delta \to X$.

Compact Great Picard

A first consequence of Kwack's extension Theorem is the following.

Theorem (Compact Great Picard)

Let X be a compact complex manifold. If X is Kobayashi hyperbolic (or equivalently Brody hyperbolic), then any map $f : \Delta^* \to X$ extends to a map $\tilde{f} : \Delta \to X$.

Proof: Take the Kobayashi distance d_X on X. Then $f : (\Delta^*, d_{\Delta^*}) \to (X, d_X)$ is distance decreasing.

General Great Picard

This argument can be generalized to obtain.

Theorem

Let X be a compact complex manifold. Let $U \subset X$ be a dense open subset and suppose that U is hyperbolically embedded in X. Then, any holomorphic map $f : \Delta^* \to U$ extends to a holomorphic map $\tilde{f} : \Delta \to X$.

For $U = \mathbb{P}^1 \setminus \{0, 1, \infty\} \subset \mathbb{P}^1$ we recover the classical Great Picard.

Proof

Let $f : \Delta^* \to U$ be holomorphic. Let h be a hermitian metric on X. By definition, there exists $\varepsilon > 0$ such that

$$\varepsilon \| \cdot \|_h |_U \leqslant F_U.$$

Denote by d_h the distance on X given by

$$d_{\varepsilon,h}(x,y) = \varepsilon \inf_{\gamma} \int_0^1 \|\gamma'(t)\|_h dt$$

where $\gamma : [0,1] \to X$ is smooth such that $\gamma(0) = x$ and $\gamma(1) = y$. By Royden's Theorem, one has

$$d_{\varepsilon,h}(x,y) \leqslant d_U(x,y) \ \ \forall\, x, y \in U.$$

Therefore, for any $z, w \in \Delta^*$

$$d_{\varepsilon,h}\big(f(z), f(w)\big) \leqslant d_{\Delta^*}(z,w).$$

Hence one can apply Kwack's extension result.

A generalization

One has the following generalization.

Theorem (Kiernan)

Let X be a compact complex space and let $U \subset X$ be a dense open subset which is hyperbolically embedded in X. Let Z be a complex manifold and let $D \subset Z$ be a simple normal crossing divisor. Then every holomorphic map $f : Z \setminus D \to U$ extends to a holomorphic map $\tilde{f} : Z \to X$.

Algebraization

Kiernan's result can be thought of as an algebraization result.

Theorem

Suppose X is a projective manifold and U a Zariski open subset hyperbolically embedded in X. Let Y be a quasi-projective variety. Then any holomorphic map $f : Y \to U$ is algebraic.

Proof: Take a compactification $\overline{Y}$ of Y such that $D = \overline{Y} \setminus Y$ is a simple normal crossing divisor. The theorem implies that f extends to $\overline{f} : \overline{Y} \to X$. By Chow, $\overline{f}$ is algebraic, therefore f is also algebraic.

Borel Hyperbolicity

The following definition is due to Javanpeykar and Kucharczyk

Definition

Let X be a quasi projective variety. X is *Borel hyperbolic* if for any quasi-projective variety Y any holomorphic map $f : Y \to X$ is algebraic.

Kiernan's theorem therefore implies that if X can be hyperbolically embedded in some projective variety, then X is Borel hyperbolic.

Picard hyperbolicity

The following definition is due to Ya Deng.

Definition

Let X be a quasi projective variety. X is *Picard hyperbolic* there exists a projective compactification $\overline{X}$ of X such that any holomorphic map $f : \Delta^* \to X$ extend to a holomorphic map $\overline{f} : \Delta \to \overline{X}$.

We have therefore proven that if X can be hyperbolically embedded in some projective variety, then X is Picard hyperbolic.

Picard imply Borel

One has

Theorem (Javanpeykar, Kucharczyk / Deng)

Let X be a quasi projective variety. If X is Picard hyperbolic, then X is Borel hyperbolic.

We will outline Deng's proof which relies on the following consequence of a deep extension result of Siu.

Theorem (Siu)

Let X be an Zariski open subset in a projective variety $\overline{X}$. If X is Picard hyperbolic, then for every $p, q \geqslant 0$, any holomorphic map $f : \Delta^p \times (\Delta^*)^q \to X$ extends to a meromorphic map $\overline{f} : \Delta^{p+q} \dashrightarrow \overline{X}$

Proof of Picard implies Brody

Let X, Y be quasi-projective varieties such that X is Picard hyperbolic. Let $f : X \to Y$ be holomorphic. Let $\overline{X}$ be a projective compactification of X such that $\overline{X} \setminus X$ is a simple normal crossing divisor. Let $\overline{Y}$ be a projective compactification of Y. The previous proposition implies that f extends to a meromorphic map $\overline{f} : \overline{X} \dashrightarrow \overline{Y}$. By Chow's result (applied to the graph of $\overline{f}$), this implies that f is algebraic.

Examples

So far our exemples of hyperbolic manifolds are

1. Hyperbolic Riemann surfaces (e.g. compact curves of genus $\geqslant 2$, Δ $\mathbb{P}^1 \setminus \{0, 1, \infty\}$).
2. Varieties with a metric with negative holomorphic sectional curvature.
3. Products and finite covers of the previous.
4. Complements of sufficiently general hyperplanes in $\mathbb{P}^n$.