Compactifications of Manifolds

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Overview

- Historical backgrounds of manifold completion
- $\mathcal{Z}$-compactification
- Pseudo-collarable manifold
- The relationship between $\mathcal{Z}$-compactification and pseudo-collarability

Manifolds with Boundary

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- Manifold with noncompact bdry

Completion of Manifolds (a.k.a. collaring problem, missing boundary problem)

A manifold M with (possibly empty) boundary is *completable* if $\exists$ a compact manifold $\widehat{M}$ with boundary and a compactum $C \subseteq \partial\widehat{M}$ such that $\widehat{M} \backslash C$ is homeomorphic to M. $\widehat{M}$ is called a *completion* of M.

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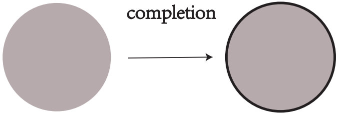

2D Manifold Completion Theorem

Theorem (G & Guilbault 2020)

A connected 2-manifold M^2 is completable iff $H_1(M^2)$ is finitely generated.

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This is mainly based on classical work of Kerékjártó (1923) and Richards (1963).

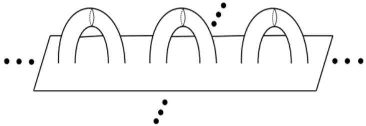
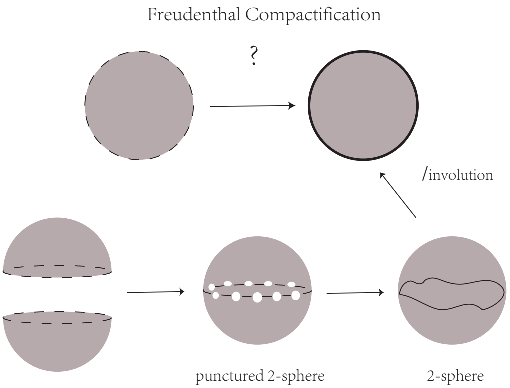


Figure: Infinite Loch Ness Monster

Kerékjártó-Freudenthal Compactification



Mine Fields in Higher Dimensions

Direct generalizations of Jordan curve theorem is unavailable.

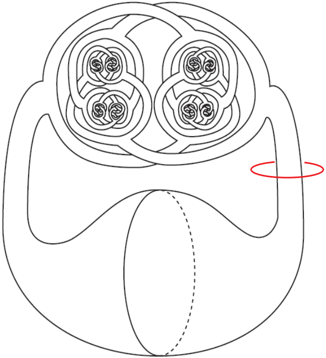


Figure: Alexander Horned Sphere

The existence of exotic open contractible 3-manifolds.

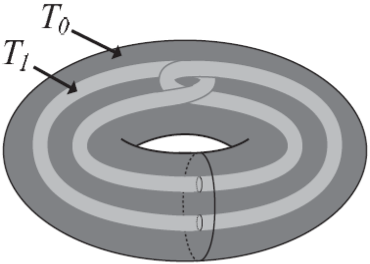


Figure: The first two stages of constructing the Whitehead Manifold

Fundamental Group at Infinity

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Fundamental Group at Infinity

Why is the Whitehead manifold not completable, i.e. not homeomorphic to $\mathbb{R}^3$?
Answer: Because it's not simply connected at infinity.

Definition

A space X is said to be *simply connected at infinity* if for all compact subsets C of X, $\exists$ a compact set $D \supset C$ in X so that the induced map $\pi_1(X \backslash D) \to \pi_1(X \backslash C)$ is trivial.

Characterization of Euclidean Spaces

Theorem (Stallings 1962, Luft '67, Freedman '86, Edwards '63, Perelman 2006)

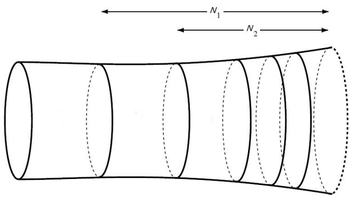
A contractible open n-manifold ($n \geq 3$) is homeomorphic to $\mathbb{R}^n$ iff it is simply connected at infinity.

Characterization of Completable 3-Manifolds

Theorem (Tucker 1974)

A 3-manifold M^3 is completable iff each component of each clean neighborhood of infinity has finitely generated π_1, modulo the Poincaré Conjecture.

Independently, Husch-Price, Finding boundary for a 3-manifold, Ann. of Math. 91 (1970), 223-235. Kakimizu, Finding boundary for the semistable ends of 3-manifolds, Hiroshima Math. J. 17 (1987), 395-403.



Characterization of Completable 1-ended Open Manifolds dim ≥ 6

Theorem (Browder, Levine and Livesay 1965)

Let M^m be a 1-ended open manifold ($m \geq 6$) that is simply connected at infinity. Then M^m is completable iff $H_*(M;\mathbb{Z})$ is finitely generated.

Characterization of Completable Manifolds dim ≥ 6

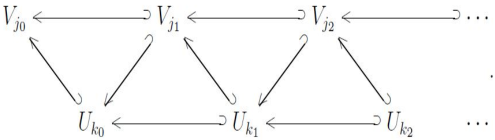
Theorem (Siebenmann 1965)

An m-manifold M^m ($m \geq 6$) with compact (possibly empty) boundary is completable iff

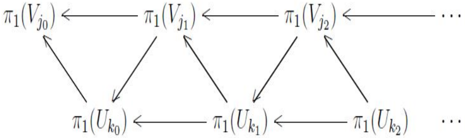
1. pro-π_1 *stable at each end of* M^m
2. M^m *is inward tame*
3. *Wall obstruction* $\sigma_\infty(M^m) = 0$.

Topological Justification of the pro-isomorphism Relation

Let $U_0 \hookleftarrow U_1 \hookleftarrow \cdots$ and $V_0 \hookleftarrow V_1 \hookleftarrow \cdots$ be two cofinal sequences of connected neighborhoods of infinity for a 1-ended space X. By going out sufficiently far in the second sequence, one arrives at a V_{j_1} contained in U_{k_0}. Similarly, going out sufficiently far in the initial sequence produces a $U_{k_0} \subseteq V_{j_0}$. Alternating back and forth produces a ladder diagram of inclusions



Applying the fundamental group functor to that diagram (ignoring base points) results in a diagram



The inverse sequences $\pi_1(U_0) \leftarrow \pi_1(U_1) \leftarrow \cdots$ and $\pi_1(V_0) \leftarrow \pi_1(V_1) \leftarrow \cdots$ are *pro-isomorphic*.

Properties of pro-isomorphism classes of Inverse Sequences of Groups

Let $G_0 \xleftarrow{\lambda_1} G_1 \xleftarrow{\lambda_2} G_3 \xleftarrow{\lambda_3} \cdots$ be an inverse sequence of groups. We say that $\{G_i, \lambda_i\}$ is

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- *stable* if it is pro-isomorphic to an inverse sequence $\{H_i, \mu_i\}$ where each μ_i is an isomorphism, or equivalently, a constant inverse sequence $\{H, \mathrm{Id}_H\}$

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- *pro-monomorphic* if it is pro-isomorphic to an inverse sequence $\{H_i, \mu_i\}$ where each μ_i is an monomorphism.

Characterization of 1-ended Completable Manifolds w/ noncompact Bdry

Theorem (O'Brien 1983)

Suppose M^m is an m-manifold ($m \geq 6$) and both M^m and ∂M^m are 1-ended. M^m is completable iff

1. *M^m is peripherally π_1-stable at infinity,*
2. *M^m is inward tame,*
3. $\sigma_\infty(M^m) = 0$,
4. *Whitehead torsion $\tau_\infty(M^m) = 0$.*

Manifold Completion Theorem

Theorem (G & Guilbault 2020)

An m-manifold M^m $(m \neq 4, 5)$ is completable iff

1. M^m *is peripherally* π_1*-stable at infinity,*
2. M^m *is inward tame,*
3. $\sigma_\infty(M^m) = 0$,
4. $\tau_\infty(M^m) = 0$.

Figure: G(L), Siebenmann(M), Guilbault(R) 2018 at Auburn Univ.

What's Next?

Question (Weinberger 1994)

Is there any kind of theory of nontame ends, i.e., manifolds with nonstable π_1 *at infinity?*

$\mathcal{Z}$-Compactification

A compactification $\widehat{X} = X \sqcup Z$ of a space X is a *$\mathcal{Z}$-compactification* if, for every open set $U \subseteq \widehat{X}$, $U \backslash Z \hookrightarrow U$ is a homotopy equivalence. Z is called a $\mathcal{Z}$-set.

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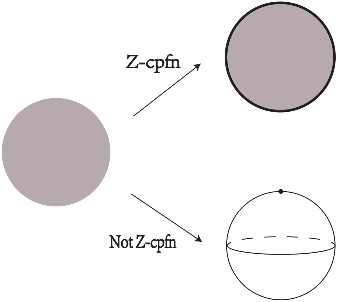
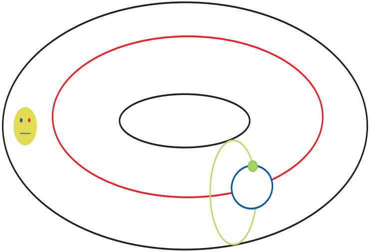
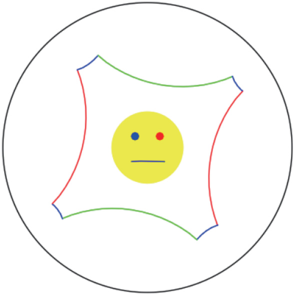
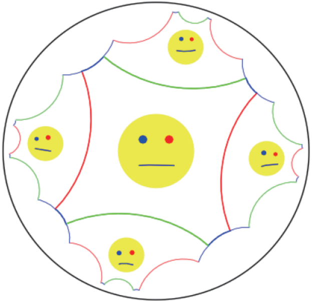
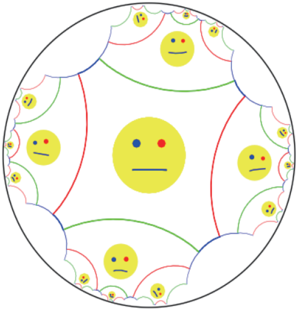


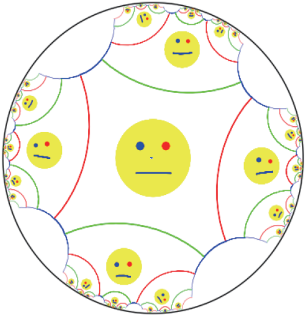
Universal Covering Space of a Punctured Torus

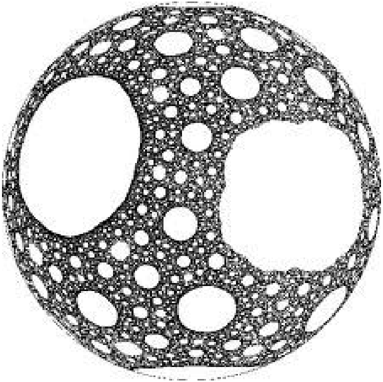












Applications of $\mathcal{Z}$-Compactifications

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Question (Guilbault 2016)

Must a closed, aspherical n-manifold ($n \geq 4$) have pseudo-collarable universal cover?

Pseudo-collarable Manifolds

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Pseudo-collarable Manifolds

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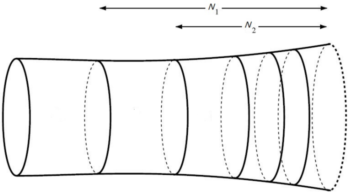
A manifold nbhd of infinity N in a manifold M is a *homotopy collar* provided $\operatorname{Fr} N \hookrightarrow N$ is a homotopy equivalence. A *pseudo-collar* is a homotopy collar which contains arbitrarily small homotopy collar nbhds of infinity.



Definition

A manifold is *pseudo-collarable* if it contains a pseudo-collar nbhd of infinity.

Characterization of Pseudo-collarable Manifolds with Bdry

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Dimension ≥ 6, Guilbault-Tinsley (2000-2003) provided a full characterization of pseudo-collarable manifolds with **compact** bdry.

Theorem (G, 2020)

An m-manifold M^m $(m \geq 6)$ is pseudo-collarable iff each of the following conditions holds:

(a) *M^m is peripherally perfectly semistable at infinity,*

(b) *M^m is inward tame,*

(c) $\sigma_\infty(M^m) = 0$.

Perfect Semistability

A *commutator* element of a group H is an element of the form $x^{-1}y^{-1}xy$ where $x, y \in H$; and the *commutator subgroup* of H; denoted $[H, H]$, is the subgroup generated by all of its commutators. The group H is *perfect* if $H = [H, H]$. An inverse sequence of groups is *perfectly semistable* if it is pro-isomorphic to an inverse sequence

$$G_0 \overset{\lambda_1}{\twoheadleftarrow} G_1 \overset{\lambda_2}{\twoheadleftarrow} G_2 \overset{\lambda_3}{\twoheadleftarrow} \cdots$$

of finitely generated groups and surjections where each $\ker(\lambda_i)$ is perfect.

Manifold Completion Theorem and Characterization of Pseudo-collarable Manifolds in dim $= 4, 5$

- In dim $= 5$, both theorems are true in TOP provided π_1 at infinity is "good" in the sense of Freedman-Quinn.

Manifold Completion Theorem and Characterization of Pseudo-collarable Manifolds in dim $= 4, 5$

- In dim $= 5$, both theorems are true in TOP provided π_1 at infinity is "good" in the sense of Freedman-Quinn.
- Both theorems fail in dim $= 4$. Counterexamples are constructed by Weinberger '87 and Kwasik-Schultz '88, independently.

Relationships among Completable, Pseudo-collarable & $\mathcal{Z}$-cpfbl Manifolds

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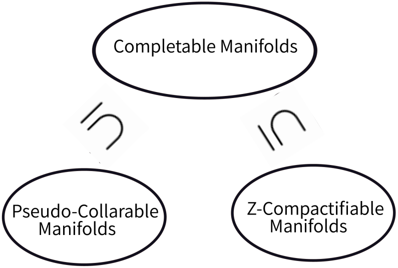

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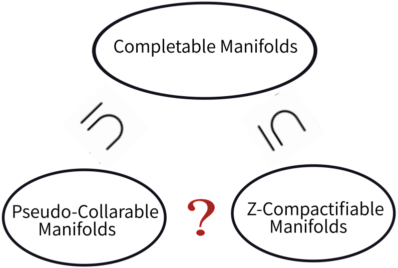

$\mathcal{Z}$-compactifiability $\implies$ Pseudo-collarability?

Davis' manifolds are both pseudo-collarable and $\mathcal{Z}$-compactifiable.
Ancel-Siebenmann 1985, Fischer 2003.

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Question (Guilbault-Tinsley 2003)

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Answer: No

Theorem (G, 2021)

There exists infinitely many contractible n-manifold M^n $(n \geq 4)$ with boundary such that M^n is $\mathcal{Z}$-cpfbl but not pseudo-collarable.

Ingredient 1: Characterization of Pseudo-collarable Manifolds with Bdry

Theorem (G, 2020)

An m-manifold M^m $(m \geq 6)$ is pseudo-collarable iff each of the following conditions holds:

(a) M^m ***is peripherally perfectly semistable at infinity***,

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(c) $\sigma_\infty(M^m) = 0$.

Ingredient 2: Bing's manifold

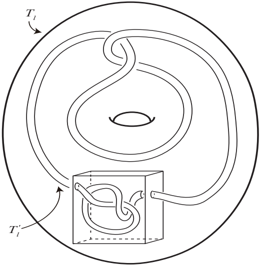

Figure: Two stages of the construction of Bing's manifold

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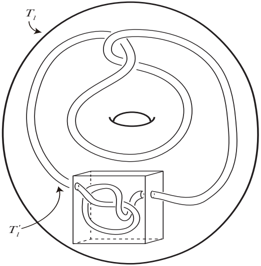


Figure: Two stages of the construction of Bing's manifold

Theorem (G, 2021)

W^3 embeds as an open subset in no compact, locally connected, locally 1-connected metric 3-space. In particular, W^3 embeds in no compact 3-manifolds.

Ingredient 3: Hypoabelian Groups

Definition

A group G is said to be *hypoabelian* if the following equivalent conditions are satisfied:

- G contains no nontrivial perfect subgroup.
- The transfinite derived series terminates at the identity.

Example

1. Abelian group
2. Solvable groups, residually solvable groups and free groups
3. Right-angled Artin group
4. The Baumslag-Solitar groups $\mathrm{BS}(1, n)$ are solvable, thus, hypoabelian.
5. Fibered knot group

Hypoabelianity is Closed under Several Algebraic Operations

1. Free products of hypoabelian groups are hypoabelian.
2. Every extension of a hypoabelian group by a hypoabelian group is hypoabelian.
3. Split amalgamated free products of hypoabelian groups are hypoabelian. Howie 1979

Hereditary of Hypoabelianity under Several Knot Operations

Lemma (G, 2021)

1. *The group of the link complement of the Whitehead link is hypoabelian.*
2. *Let K_C be a non-trivial knot, K_W be a satellite knot and (V_P, K_P) be the pattern. Suppose the knot group of K_C and $\pi_1(V_P \backslash K_P)$ are hypoabelian, and that the Alexander polynomial of K_W is nontrivial. Then the knot group of K_W is hypoabelian.*
3. *Let K_1 and K_2 be knots and G_1 and G_2 be the corresponding knot groups. If G_i is hypoabelian, then the knot group of $K_1 \# K_2$ is hypoabelian.*

Hypoabelianity and Perfect Semistability

Lemma (Guilbault-Tinsley, 2003)

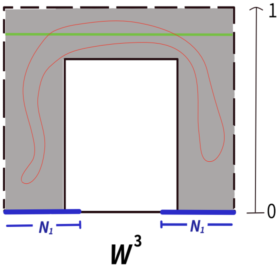
Let

$$G_0 \leftarrow G_1 \leftarrow G_2 \leftarrow \cdots$$

be an inverse sequence of groups with surjective but non-injective bonding homomorphism. Suppose each G_i is a hypoabelian group. Then the inverse sequence is not perfectly semistable.

$W^3 \times [0,1)$ is a counterexample

First we show that $W^3 \times [0,1)$ admits a $\mathcal{Z}$-compactification. This follows from a result of Bestvina-Mess, The boundary of negatively curved groups, J. Amer. Math. Soc. 4 (1991), no. 3, 469-481.



Second, by Guilbault-Tinsley's lemma and the characterization of pseudo-collarable manifolds, it suffices to show that the fundamental group of the end of $W^3 \times [0, 1)$ is isomorphic to a sequence of hypoabelian groups. In our case, π_1 at the end is isomorphic to an inverse sequence of "knot groups". That is, let K be a trefoil knot and G be the knot group $\pi_1(S^3 \backslash K)$; let K^{Wh} be a twisted Whitehead double of K and G^{Wh} be the knot group of K^{Wh}, π_1 at the end is isomorphic to

$$G \leftarrow G^{Wh} *_{\mathbb{Z}} G \leftarrow (G^{Wh} *_{\mathbb{Z}} G)^{Wh} *_{\mathbb{Z}} G \leftarrow \cdots$$

Applying the lemmas regarding the hereditary of hypoabelianity under knot operations to complete the proof.

Question

Does there exist an open manifold which is $\mathcal{Z}$-compactifiable but not pseudo-collarable?

A Big Open Question

Question (Chapman-Siebenmann'76)

If an m-manifold M^m ($m \geq 6$) satisfies

1. *M^m is inward tame,*
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Theorem (G & Guilbault, 2020)

Let M^m be an m-manifold ($m \geq 5$). M^m satisfies Conditions (1)-(3) iff $M^m \times [0,1]$ is $\mathcal{Z}$-cpfbl.

Pseudo-collarability $\implies$ $\mathcal{Z}$-compactifiability?

Question

Does Pseudo-collarability imply $\mathcal{Z}$-compactifiability?

Pseudo-collarability $\Longrightarrow$ $\mathcal{Z}$-compactifiability?

Question

Does Pseudo-collarability imply $\mathcal{Z}$-compactifiability?

Answer: No

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Answer: No

Theorem (G. in progress)

There exists an n-manifold M^n $(n \geq 4)$ satisfying Conditions (1)-(3) but not admitting a $\mathcal{Z}$-compactification. In particular, M can be pseudo-collarable.

Thank you for your attention.

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