② $s_1, s_2, \cdots, s_N \in H^0(X, mL)$

define $\varphi_\alpha = \log\left(\sum_{i=1}^{N} |s_{i,\alpha}|^{2/m}\right)^{1/2}$, then $\varphi_\alpha = \varphi_\beta + \log|g_{\alpha\beta}|$, $\varphi_\alpha \in psh(V_\alpha)$.

③ Let $s_1^{(m)}, \cdots s_{N_m}^{(m)} \in H^0(X, mL)$, $m \in \mathbb{N}$

define $\varphi^{Siu}_{min,\alpha} = \log\left(\sum_{m=1}^{\infty} \theta_m \sum_{k=1}^{N_m} |s_{k,\alpha}|^{2/m}\right)^{1/2}$, where $\{\theta_m > 0\}$ is a suitable sequence s.t. $\sum_{m=1}^{\infty} \theta_m \cdot \sum \cdots$ converges.

<u>Rmk./Exer.*</u>:

Let $X/\mathbb{C}$ be a proj. mfd, L a psef. line bdl on X, $\theta \in c_1(L)$ a s.m. rep.

Let $\varphi^{Demailly}_{min} = \sup\{u \mid \theta + \frac{i}{\pi}\partial\bar\partial u \geq 0$ in the sense of currents$\}$, $u \leq 0$

Let $s_1^{(m)}, \cdots, s_{N_m}^{(m)} \in H^0(X, mL)$ be the basis of $H^0(mL)$, $m \in \mathbb{N}$.

Then $R(L) = \bigoplus_{m=1}^{\infty} H^0(X, mL)$ is finitely generated $\Longleftrightarrow$ $\varphi^{Dem}_{min} = \varphi^{Siu}_{min} + O(1)$.

#

<u>Multiplier ideal sheaves</u>:

Local case. $\Omega \subseteq \mathbb{C}^n$, $\varphi \in psh(\Omega)$

Def. the multiplier ideal sheaf associated to φ is defined by

$$\mathcal{J}(\varphi) = \bigcup_{x \in \Omega} \mathcal{J}(\varphi)_x, \text{ where } \mathcal{J}(\varphi)_x = \{f \in \mathcal{O}_x \mid |f|^2 e^{-2\varphi} \in L^1_x\}.$$

Thm. $\mathcal{J}(\varphi)$ is coherent on Ω. ~~Moreover~~ Indeed, if Ω is a bounded Stein open set, then $\mathcal{J}(\varphi)$ is generated by any Hilbert basis of the L^2-space $H^2(\Omega, \varphi)$

$$= \left\{ f \in \mathcal{O}(\Omega) \,\middle|\, \int_\Omega |f|^2 e^{-2\varphi} dV < \infty \right\}.$$

Proof: The statement is local, w.l.o.g., assume Ω is an open ball in $\mathbb{C}^n$.
(or any bded. pseudoconvex open set)

$H^2(\Omega, \varphi)$ is a Hilbert space. Let $\{e_k\}_{k=1}^{\infty}$ be a basis, $\mathcal{F}^l$ = the sheaf generated by $\{e_1, e_2, \dots, e_l\}$.

Then $\mathcal{F}^l$ is coherent for any $l \in \mathbb{N}$.

$$\mathcal{F}^1 \subseteq \mathcal{F}^2 \subseteq \dots \subseteq \mathcal{F}^l \subseteq \dots \subseteq \mathcal{O}_\Omega$$

Strong Noetherian property of coherent sheaves $\Rightarrow$ on any $K \subset\subset \Omega$, the sequence is stationary for $l \geq l_0$

$\Rightarrow$ the sheaf $\mathcal{J}$ generated by $H^2(\Omega, \varphi)$ is coherent.

CLAIM: $\mathcal{J}(\varphi) = \mathcal{J}$.

It's clear that $\mathcal{J} \subseteq \mathcal{J}(\varphi)$, we need to verify that $\mathcal{J}(\varphi) \subseteq \mathcal{J}$

This follows from Nakayama's lemma:

[A: a local ring, E: a f.g. A-module, $F \subseteq E$: a submodule, $\mathfrak{m}$: maximal ideal
then $E = F + \mathfrak{m} \cdot E \Rightarrow E = F$]

In our setting let $E = \mathcal{J}(\varphi)_x$, $F = \mathcal{J}_x$, $A = \mathcal{O}_x$, $\mathfrak{m} = \mathfrak{m}_x$.

We need to verify that $\mathcal{J}(\varphi)_x = \mathcal{J}_x + \mathfrak{m}_x \cdot \mathcal{J}(\varphi)_x$

Indeed, we will prove that: $\forall s \in \mathbb{N}_{\geq 0}$, $\mathcal{J}(\varphi)_x = \mathcal{J}_x + \mathfrak{m}_x^{s+1} \mathcal{J}(\varphi)_x$.

更正：下面实际证明的是J(\phi))_x = J_x + m_x ^{s+1} \cap J(\phi)_x,
应用的交换代数结果为 Krull lemma.

Let $f \in \mathcal{J}(\varphi)_x$ in some nbhd. V of x, θ : a cutoff function with spt in V s.t. $\theta = 1$ near x

We solve the equation $\bar\partial u = \bar\partial(\theta f)$ with metric weight $\tilde\varphi = \varphi + (n+s)\log|z-x| + |z|^2$

(Hormander's L^2-existence thm)

$\Rightarrow$ the equation has a solution u satisfying:

$$\int_\Omega |u|^2 e^{-2\tilde\varphi} \leq \int_\Omega |\bar\partial(\theta f)|^2 e^{-2\tilde\varphi} < \infty$$

$$\wr\wr$$

$$\int_\Omega |u|^2 e^{-2\varphi} \cdot |z-x|^{-2(n+s)} \Rightarrow u_x \in \mathcal{J}(\varphi)_x \cdot m_x^{s+1}$$

Let $F = \theta f - u$, then $F \in H^2(\Omega, \varphi)$ and $\underset{\substack{\cap \\ \mathcal{J}(\varphi)_x}}{F_x} - \underset{\substack{\cap \\ \mathcal{J}(\varphi)_x \cdot m_x^{s+1}}}{f_x} = -u_x$

<u>Lem.</u> $F_x \in \mathcal{J}_x$.

This follows from the thm on <u>closure of modules</u>:

Any submodule $M \subseteq \mathcal{O}_x^{\oplus p}$ is closed in the following sense: if $F \in \mathcal{O}_x^{\oplus p}(V)$ and $\exists\, f_k \in \mathcal{O}^{\oplus p}(V)$ s.t. $f_k \xrightarrow{C^0_{loc}} F$ and the germ of $f_k \in M$ then the germ of $F \in M$.

Applying this to $M = \mathcal{J}$ shows that $F \in \mathcal{J}_x$. #

Exercise 1. Provide the details of the proof on the above lemma and closure of modules theorem.
(ref. [Gunning-Rossi, Analytic Functions of Several Complex Variables, chap 2, sect. E])

global case of multiplier ideal sheaves.

$(X/\mathbb{C}, L, h)$: By the def. of psef line bdl (L, h) ($\Leftrightarrow$ metric weights $\varphi_\alpha \in \mathrm{psh}(V_\alpha)$)

$\varphi_\alpha = \varphi_\beta + \log|g_{\alpha\beta}|$ on $V_\alpha \cap V_\beta$

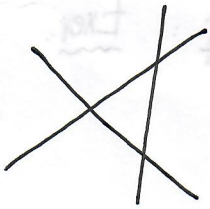
For $x \in V_\alpha \cap V_\beta$, it is clear that $\mathcal{J}(\varphi_\alpha)_x = \mathcal{J}(\varphi_\beta)_x = \mathcal{J}(h)_x$.

Def. The multiplier ideal sheaf $\mathcal{J}(h)$ associated to (L, h) is given by $\mathcal{J}(h) = \bigcup_{x \in X} \mathcal{J}(h)_x$.

Note that singular metrics can be defined on $\mathbb{R}$-line bdls by replacing $|g_{\alpha\beta}|$ be some real combination. #

Examples

① $X/\mathbb{C}$, $D = \sum_{j=1, \alpha_j > 0}^{k} \alpha_j D_j$: effective $\mathbb{R}$-divisor with SNC spt.

D: 

Associated to D this is a singular metric φ_D : locally $D_j = g_j^{-1}(0)$ for some hol. function g_j

$$\varphi_D = \sum_{j=1}^{k} \alpha_j \log|g_j|$$ — these φ_D patch to a metric weight on the $\mathbb{R}$-line bdl

$$L_D = \sum_{j=1}^{k} \alpha_j L_{D_j}.$$

Let's compute $\mathcal{J}(\varphi_D)$:

$$f \in \mathcal{J}(\varphi_D)_x \Leftrightarrow \int_{V_x} |f|^2 e^{-2\varphi_D} = \int_{V_x} |f|^2 \Big/ \prod_{j=1}^{k} |g_j|^{2\alpha_j} < \infty \quad (*)$$

D SNC $\Rightarrow$ for every x, the defining functions g_j complete to a local coordinate system $(z_1, z_2, \cdots z_n)$

Exercise 2.

Exer : $(*)$ holds $\Leftrightarrow$ f is divisible by $\prod_j g_j^{m_j}$ where $m_j - \alpha_j > -1$ (i.e., $m_j \geq \lfloor \alpha_j \rfloor$)

Therefore, $\mathcal{J}(\varphi_D) = \mathcal{O}\left(-\sum_{j=1}^{k} \lfloor \alpha_j \rfloor D_j\right) = \mathcal{O}(-\lfloor D \rfloor)$.

注记：以下关于push-forward后面的层，记号有些混乱，请自行更正。

#

How about the case when D is NOT SNC? This follows from the projection formula:

Let $\mu: \hat{X} \to X$ be a modification between complex mflds, $\varphi \in \mathrm{psh}(X)$, then

$$\mathcal{J}(\varphi) = \mu_* \mathcal{O}_{\hat{X}}\left(K_{\hat{X}/X} \otimes \mathcal{J}(\varphi \circ \mu)\right),$$ where $K_{\hat{X}/X} = K_{\hat{X}} - \mu^* K_X$ is the relative canonical sheaf.

$$\left(\Leftrightarrow K_X \otimes \mathcal{J}(\varphi) = \mu_*\left(K_{\hat{X}} \otimes \mathcal{J}(\varphi \circ \mu)\right)\right).$$

Proof: Exer. Hint: $f \in \mathcal{J}(\varphi)_x \Leftrightarrow \int_U |f|^2 e^{-2\varphi} dz \wedge d\bar{z} < \infty$, by using the identification $f(z) \mapsto f(z)\, dz$, $f \in \mathcal{J}(\varphi)_x$ can be considered as a local section of $K_X \otimes \mathcal{J}(\varphi)$

Show that $K_X \otimes \mathcal{J}(\varphi) \xrightarrow{\sim} \mu_*\left(K_{\hat{X}} \otimes \mathcal{J}(\varphi \circ \mu)\right)$ $\left(\mu: \hat{X} \to X,\ w \mapsto z = \mu(w)\right)$

$f(z)\, dz \longmapsto \mu^* f(w)\, \mathrm{Jac}(\mu) \cdot dw$, $\mathrm{Jac}(\mu) = \frac{\partial \mu}{\partial w} = \left[\frac{\partial z_k}{\partial w_l}\right]$.

Exercise 3. Provide the details for proof of the projection formula.

#

② $X/\mathbb{C}$, $D = \sum_{j=1, \alpha_j > 0}^{k} \alpha_j D_j$: effective $\mathbb{R}$-divisor. $\left(\varphi_D = \sum \alpha_j \log |g_j|\right)$

Black box : Hironaka's desingularization theorem.

Consider a log-resolution of (X, D), i.e. a modification $\mu: \hat{X} \to X$ satisfying

- $\hat{X}$ smooth
- $\mu^* D + \mathrm{Exc}(\mu)$ SNC (associated to $\mu^* D$ is $\varphi_D \circ \mu$)

By the projection formula,

$$\mathcal{J}(\varphi_D) = \mu_*\mathcal{O}_{\hat{X}}(K_{\hat{X}/X} \otimes \mathcal{J}(\varphi_D \circ \mu))$$

$$= \mu_*\mathcal{O}_{\hat{X}}(K_{\hat{X}/X} - \lfloor \mu^* D \rfloor).$$

<u>Rmk.</u> On the right hand side, μ can be different, but $\mathcal{J}(\varphi_D)$ is fixed.

- analytic multiplier ideal sheaf $\leftrightarrow$ algebraic multiplier ideal sheaf.

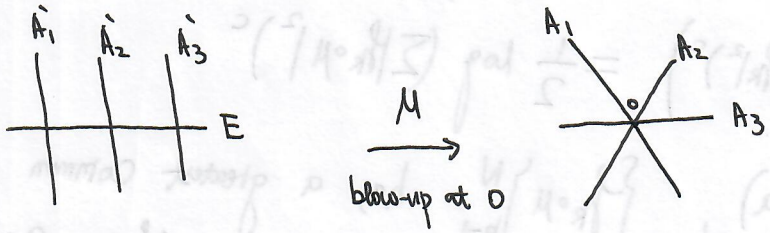
<u>Def.</u> (PIAG, vol 2, chap. 9)

$X/\mathbb{C}$: ~~proj. mfd~~ s.m. complex var., D : effective $\mathbb{Q}$-divisor, $\mu : \hat{X} \to X$: a log resolution of (X, D)

then $\mathcal{J}(D)$ is defined by $\mathcal{J}(D) = \mu_*\mathcal{O}_{\hat{X}}(K_{\hat{X}/X} - \lfloor \mu^* D \rfloor)$

(In particular, $\mathcal{J}(\varphi_D) = \mathcal{J}(X, D)$.)

E.g. $X = \mathbb{C}^2$, $D = \frac{2}{3}(A_1 + A_2 + A_3)$, where A_1, A_2, A_3 are 3 distinct lines through 0.



$\mu^*(A_1 + A_2 + A_3) = A_1' + A_2' + A_3' + 3E$, where A_i' is the proper transform of A_i.

Thus, $\mu^* D = \frac{2}{3}(A_1' + A_2' + A_3') + 2E$

Note that $K_{\hat{X}/X} = E$, we get $\mathcal{J}(X, D) = \mathcal{J}(\varphi_D) = \mu_*\mathcal{O}_{\hat{X}}(-E)$

maximal ideal of hol. functions vanishing at 0.

More examples:

$\mathfrak{a} \subseteq \mathcal{O}_X$: an ideal sheaf, locally $\mathfrak{a} = \langle f_1, f_2, \dots, f_N \rangle$.

For $c > 0$, we define $\varphi_{\mathfrak{a}^c} = \frac{1}{2} \log \left(|f_1|^2 + \dots + |f_N|^2 \right)^c$

Rmk./Exer. If $g_1, \dots g_M$ is another family of local generators of $\mathfrak{a}$, then we get another psh function $\tilde{\varphi}_{\mathfrak{a}^c} = \frac{1}{2} \log \left(\sum_{k=1}^{M} |g_k|^2 \right)^c$, we have $\tilde{\varphi}_{\mathfrak{a}^c} = \varphi_{\mathfrak{a}^c} + O(1)$.

Therefore $\mathcal{J}(\varphi_{\mathfrak{a}^c}) = \mathcal{J}(\tilde{\varphi}_{\mathfrak{a}^c})$.

algebro-geometrical def: Let $\mu : \hat{X} \to X$ be a log resolution of $(X, \mathfrak{a})$, i.e. a birational morphism satisfying : • $\hat{X}$ sm. • $\mu^* \mathfrak{a} = \mathfrak{a} \cdot \mathcal{O}_{\hat{X}} = \mathcal{O}(-F)$ with $F + \mathrm{Exc}(\mu)$ SNC

$$\mathcal{J}(X, \mathfrak{a}^c) := \mu_* \mathcal{O}_{\hat{X}}(K_{\hat{X}/X} - \lfloor cF \rfloor).$$

CLAIM: $\mathcal{J}(X, \mathfrak{a}^c) = \mathcal{J}(\varphi_{\mathfrak{a}^c})$

Exercise 4. Prove the claim. (hint: apply the projection formula to analytic multiplier ideal sheaf and the property of log-resolution.)

· Lelong numbers and $\mathcal{J}(\varphi)$.

Recall, given $\varphi \in \mathrm{psh}(\Omega)$, $x \in \Omega$, $\nu(\varphi, x) = \sup\{t \geq 0 \mid \varphi(z) \leq t \log|z-x| + O(1)$ around $x\}$

$\Rightarrow$ Prop./Exer.: If $\nu(\varphi, x) \geq n+s$ for some $s \in \mathbb{Z}_{\geq 0}$, then $\mathcal{J}(\varphi)_x \subseteq m_x^{s+1}$.

On the other hand,

Thm (Skoda 1972) if $\nu(\varphi, x) < 1$, then $e^{-2\varphi} \in L^1_x$, in particular, $\mathcal{J}(\varphi)_x = \mathcal{O}_x$.

#

Denote $E_c(\varphi) = \{x \in \Omega \mid \nu(\varphi, x) \geq c\}$, $c > 0$

$V(\mathcal{J}(\varphi))$ = the zero variety of $\mathcal{J}(\varphi) = \{x \in \Omega \mid e^{-2\varphi} \notin L^1_x\}$

then we get: $E_n(\varphi) \subseteq V(\mathcal{J}(\varphi)) \subseteq E_1(\varphi)$.

Thm. (Boucksom-Favre-Jonsson) + (Guan-Zhou's openness thm)

Let $\varphi_1, \varphi_2 \in \mathrm{psh}(\Omega)$, $x \in \Omega$, ~~then~~: ~~$\pi: \hat{\Omega} \to \Omega$ blow up at x.~~

TFAE:

- $\forall t > 0$, $\mathcal{J}(t\varphi_1)_x = \mathcal{J}(t\varphi_2)_x$

- $\forall$ proper modification $\pi: \Omega_\pi \to \Omega$ over x and any p.t. $p \in \pi^{-1}(x)$

$$\nu(\varphi_1 \circ \pi, p) = \nu(\varphi_2 \circ \pi, p).$$

#