Lelong numbers

FACT / Exer: $S \subseteq \mathbb{R}^n$ a convex open set, then u is convex on S

Exercise 1.

$$\iff \quad U(z) = u(x+iy) := u(x) \in \mathrm{Psh}(S + i\mathbb{R}^n).$$

Cor: $\Omega \subseteq \mathbb{C}^n$, $a \in \Omega$, $u \in \mathrm{Psh}(\Omega)$, then $r \mapsto \sup\limits_{B(a,r)} u$ is convex w.r.t. $\log r$.
(three-circle thm)

/ write $f(t) = \sup\limits_{B(a,e^t)} u$, $w = t + is \in \mathbb{C}$, then $f(t) = \sup\limits_{\zeta \in B(0,1)} u(a + e^{w} \cdot \zeta) := F(w)$

It's easy to check that $F(w)$ is psh w.r.t. w, and indp. of $\mathrm{Im}\, w$.

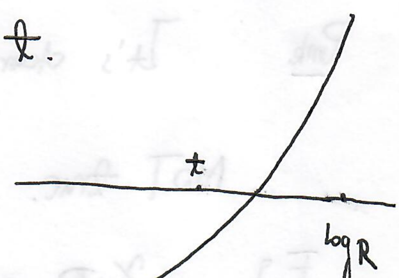
$\Rightarrow f(t)$ is convex w.r.t. t. #

Rmk./Exer: The same convexity also holds for $r \mapsto \fint_{B(a,r)} u$, $r \mapsto \fint_{S(a,r)} u$, $r \mapsto u * \rho_r(a)$

Setting as in the Cor., assume $B(a, R_0) \subset\subset \Omega$, by Cor,

$$f(t) = \sup_{B(a, e^t)} u, \quad t \in (-\infty, \log R] \text{ is convex (and increasing w.r.t. } t)$$

$$\Rightarrow \frac{f(t) - f(\log R)}{t - \log R} \ (\geq 0) \text{ is non-decreasing w.r.t. } t.$$



$$\Rightarrow \lim_{t \to -\infty} \frac{f(t) - f(\log R)}{t - \log R} := f'(-\infty) \text{ exists.}$$

$\uparrow$ Def. the Lelong number of u at a.

In the other words,

$$\nu_a(u) = f'(-\infty) = \lim_{t\to 0} \frac{\sup_{B(a,t)} u}{\log t}.$$

By convexity of $f(t)$ again,

$$f(t) - f(\log R) \leq (t - \log R)\, f'(-\infty) = (t - \log R)\, \nu_a(u), \quad t \leq \log R$$

$$\Leftrightarrow \quad \cancel{u(z)} \sup_{B(a,e^t)} u \leq \sup_{B(a,R)} u + \nu_a(u)\cdot(t - \log R)$$

For $z \in B(a,R)$, write $t = \log|z-a|$, we get $u(z) \leq \nu_a(u) \log\frac{|z-a|}{R} + \sup_{B(a,R)} u$.

Exercise 2.

prop. / Exer. $\nu_a(u) = \max\left\{ C \geq 0 \;\middle|\; u(z) \leq C\log|z-a| + O(1) \text{ near } a \right\}$.

Rmk. $\nu_a(u)$ can be equivalently characterized by $\fint_{B(a,r)} u$, $\fint_{S(a,r)} u$, $u * P_r(a)$.

Exercise 3.

Examples: / Exer:

Let $f \in \mathcal{O}(\Omega)$, $u = \log|f|$, then $\nu_a(u) = \mathrm{ord}_a(f) = \max\left\{ m \in \mathbb{N} \;\middle|\; D^\alpha f(a) = 0 \text{ for any } |\alpha| < m \right\}$

Rmk. It's clear that $\nu_a(u) > 0 \Rightarrow u(a) = -\infty$, however, the converse is NOT true.

E.g. $\chi: \mathbb{R}_- \to \mathbb{R}$ convex, increasing, consider $u(z) = \chi(\log|z|)$ for suitable χ $\quad \nu_0(u) = \chi'(-\infty)$.

for example, $\chi(t) = -\log(-t)$ or $(-t)^\alpha$, $(0<\alpha<1)$ #

Lelong numbers for positive (d-closed) (1,1)-currents. (locally, $T = \frac{i}{\pi}\partial\bar\partial u$, $u \in psh$)

$\Omega \subseteq \mathbb{C}^n$, T : d-closed positive (1,1)-current on Ω, $\omega = \frac{i}{2}\partial\bar\partial|z|^2$: Euclidean metric on Ω

$\sigma_T := T \wedge \frac{\omega^{n-1}}{(n-1)!}$ — the trace measure of T

$H \subseteq \mathbb{C}^n$: linear subspace of $\dim_{\mathbb{C}} = n-1$

Given $a \in \Omega$, FACT: $\frac{\sigma_T(B(a,r))}{\sigma_{[H]}(B(a,r))} = \int_{B(a,r)} T \wedge \frac{\omega^{n-1}}{(n-1)!} \Big/ \int_{B(a,r)\cap H} \frac{\omega^{n-1}}{(n-1)!}$ is non-decreasing w.r.t. $r > 0$.

Thm. $\lim\limits_{r\to 0} \downarrow \frac{\sigma_T(B(a,r))}{\sigma_H(B(a,r))} := \nu(T, a) = \nu_a(u)$. #

E.g. $T = [Z_f] = \sum m_j [Z_j]$, where $Z_f = \mathrm{div}(f)$ for some non-zero hol. function f

then $\nu(T, a) = \mathrm{mult}_a Z_f = \mathrm{ord}_a(f)$.

Singular metrics

$X/_{\mathbb{C}}$: complex mfd, L : hol. line bdl on X, i.e. $L \in H^1(X, \mathcal{O}^*)$

is given by the data: $X = \bigcup\limits_\alpha V_\alpha$, $g_{\alpha\beta} \in \mathcal{O}^*(V_\alpha \cap V_\beta)$ satisfying

$g_{\alpha\beta} \circ g_{\beta\gamma} \cdot g_{\gamma\alpha} = \mathrm{id}$

$g_{\alpha\beta}^{-1} = g_{\beta\alpha}$

a hol section $S = \{S_\alpha\}$ is given by $S_\alpha \in \mathcal{O}(V_\alpha)$ satisfying $S_\alpha = g_{\alpha\beta} S_\beta$

a smooth metric h on L is given by $0 < h_\alpha \in C^\infty(V_\alpha)$ s.t. $|S_\alpha|^2 h_\alpha = |S_\beta|^2 h_\beta$

$\Leftrightarrow$ Write $h_\alpha = e^{-2\varphi_\alpha}$ for some $\varphi_\alpha \in C^\infty(V_\alpha)$ (metric weight)   i.e. $h_\alpha = |g_{\alpha\beta}|^{-2} h_\beta$

a s.m. metric on L is given by $\varphi_\alpha \in C^\infty(V_\alpha)$ with $\varphi_\alpha = \varphi_\beta + \log|g_{\alpha\beta}|$

(L, h) Hermitian line bdl $\rightsquigarrow$ Chern connection D $\rightsquigarrow$ Chern curvature

$$\Theta_{L,h} = D^2 \overset{\text{loc.}}{=} \bar\partial(h^{-1}\partial h) = -\partial\bar\partial \log h = 2\partial\bar\partial\varphi$$

Def. $c_1(L) = \{\frac{i}{2\pi} D^2\} = \{\frac{i}{2\pi}\Theta_{L,h}\}$ / locally $\frac{i}{2\pi}\Theta_{L,h} = \frac{i}{\pi}\partial\bar\partial\varphi$.

$$\in H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{Z})$$

Exercise 4. (to make it more precise, we assume that X is Kaehler or c_1(L) is a Bott-Chern class)

Rmk./Exer. Fix a s.m. rep. $\theta \in c_1(L)$, then for any other s.m. rep $\alpha \in c_1(L)$ $(\alpha = \theta + \frac{i}{\pi}\partial\bar\partial f)$, $\exists$ a metric $\tilde h$ s.t. $\alpha = \frac{i}{2\pi}\Theta_{\tilde h}$. ($\tilde h$ can be defined by h and f).

Def. a singular metric h on L is a collection of functions $0 < h_\alpha = e^{-2\varphi_\alpha}$ s.t. ~~(weight)~~

$$\varphi_\alpha \in L^1_{loc}(V_\alpha), \quad \varphi_\alpha = \varphi_\beta + \log|g_{\alpha\beta}|$$

$\frac{i}{2\pi}\Theta_{L,h} \overset{\text{loc}}{=} \frac{i}{\pi}\partial\bar\partial\varphi_\alpha$ is called the curvature current of (L, h).
(globally defined)

$X/\mathbb{C}$: cpt complex mfd., ω: a fixed Hermitian metric.

Def L is called <u>psef.</u> if $\exists$ a singular metric h s.t. $\frac{i}{2\pi}\Theta_{L,h} \geq 0$ in the sense of currents

(i.e. the metric weight $\varphi_\alpha \in psh(V_\alpha)$)

<u>big</u> if $\frac{i}{2\pi}\Theta_{L,h} \geq \delta\omega$ currents.

<u>ample</u> if $\exists$ a sm. metric s.t. $\frac{i}{2\pi}\Theta_{L,h} > 0$ everywhere ($\Leftrightarrow \frac{i}{2\pi}\Theta_{L,h} \geq \delta\omega$ for some $\delta > 0$)

<u>nef.</u> if $\forall\ \varepsilon > 0$, $\exists$ a s.m. metric h_ε, s.t. $\frac{i}{2\pi}\Theta_{L,h_\varepsilon} > -\varepsilon\omega$.

(The equivalence with algebro-geometric setting will be discussed later.)

Exer. Assume L is psef, then $\forall$ positive $(1,1)$-current $T \in c_1(L)$, $\exists$ a singular metric h s.t. $T = \frac{i}{2\pi}\Theta_{L,h}$.

<u>Examples of singular metrics</u>

$(X/\mathbb{C}, L)$

① $0 \not\equiv s \in H^0(X, mL)$ for some $m \in \mathbb{N}$, i.e. locally $s = \{s_\alpha\}$, $s_\alpha \in \mathcal{O}(V_\alpha)$ satisfying

$$s_\alpha = g_{\alpha\beta}^m s_\beta$$

then $\varphi_\alpha := \frac{1}{m}\log|s_\alpha|$ satisfies $\varphi_\alpha = \varphi_\beta + \log|g_{\alpha\beta}|$ $\rightsquigarrow \varphi = \{\varphi_\alpha\}$

$h = \{e^{-2\varphi_\alpha}\}$

Curvature current $\frac{i}{\pi}\partial\bar\partial\varphi_\alpha = \frac{1}{m}\frac{i}{\pi}\partial\bar\partial\log|s_\alpha| = \frac{1}{m}[s_\alpha^{-1}(0)]$

In particular $\frac{i}{2\pi}\Theta_h = \frac{1}{m}[Z_s]$

② $s_1, s_2, \cdots, s_N \in H^0(X, mL)$

define $\varphi_\alpha = \log\left(\sum_{i=1}^{N} |s_{i,\alpha}|^{2/m}\right)^{1/2}$, then $\varphi_\alpha = \varphi_\beta + \log|g_{\alpha\beta}|$, $\varphi_\alpha \in psh(V_\alpha)$.

③ Let $s_1^{(m)}, \cdots s_{N_m}^{(m)} \in H^0(X, mL)$, $m \in \mathbb{N}$

define $\varphi^{Siu}_{min,\alpha} = \log\left(\sum_{m=1}^{\infty} \theta_m \sum_{k=1}^{N_m} |s_{k,\alpha}|^{2/m}\right)^{1/2}$, where $\{\theta_m > 0\}$ is a suitable sequence

s.t. $\sum_{m=1}^{\infty} \theta_m \sum \cdots$ converges.

Exercise* 5. (Ref. [Boucksom-Eyssidieux-Guedj-Zeriahi, Acta 2010, prop. 6.5])

Rmk./Exer.*:

Let $X/\mathbb{C}$ be a proj. mfd, L a psef. line bdl on X, $\theta \in c_1(L)$ a s.m. rep.

Let $\varphi^{Demailly}_{min} = \sup\{u \underset{u\le 0}{} \mid \theta + \frac{i}{\pi}\partial\bar\partial u \geq 0$ in the sense of currents$\}$

Let $s_1^{(m)}, \cdots, s_{N_m}^{(m)} \in H^0(X, mL)$ be the basis of $H^0(mL)$, $m \in \mathbb{N}$.

Then $R(L) = \bigoplus_{m=1}^{\infty} H^0(X, mL)$ is finitely generated $\Longleftrightarrow$ $\varphi^{Dem}_{min} = \varphi^{Siu}_{min} + O(1)$.

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Multiplier ideal sheaves:

Local case. $\Omega \subseteq \mathbb{C}^n$, $\varphi \in psh(\Omega)$

Def. the multiplier ideal sheaf associated to φ is defined by

$$\mathcal{J}(\varphi) = \bigcup_{x\in\Omega} \mathcal{J}(\varphi)_x, \text{ where } \mathcal{J}(\varphi)_x = \{f \in \mathcal{O}_x \mid |f|^2 e^{-2\varphi} \in L^1_x\}.$$