Invariants defined by multiplier ideal sheaves.

Def. (X, D), $D \geq 0$ $\mathbb{Q}$-divisor

- (X, D) is called k.l.t. (Kawamata log-terminal) if $\mathcal{J}(X, D) = \mathcal{J}(\varphi_D) = \mathcal{O}_X$
- ... l.c. (log canonical) if $\forall\, 0 < \varepsilon < 1$, $\mathcal{J}(X, (1-\varepsilon)D) = \mathcal{J}((1-\varepsilon)\varphi_D) = \mathcal{O}_X$.

Exercise 1.

prop./Exer: prove that (X, D) is k.l.t if $\operatorname{ord}_E (K_{\hat{X}/X} - \mu^* D) > -1$ for any prime divisor E appearing in the log resolution $\mu: \hat{X} \to X$ of (X, D).

Def. $D \geq 0$. $\mathbb{Q}$-divisor.

the log canonical threshold (l.c.t) of D at x is $\operatorname{lct}(D; x) = \inf\{c \in \mathbb{Q} \mid \mathcal{J}(X, cD)_x \subseteq \mathfrak{m}_x\}$

Exer. $= \sup\{c \in \mathbb{Q} \mid e^{-2c\varphi_D} \in L^1_x\}$

Exercise 2.

prop./Exer. If $\mu: \hat{X} \to X$ is a log resolution of (X, D) with $\mu^* D = \sum r_j E_j$, $K_{\hat{X}/X} = \sum b_j E_j$

then
$$\operatorname{lct}(D, x) = \min_{\substack{j \\ \text{s.t. } \mu(E_j) \ni x}} \left\{ \frac{b_j + 1}{r_j} \right\}$$

Exer: Let $f \in \mathcal{O}_{\mathbb{C}^n, x}$, define $C_x(f) = \sup\left\{ t > 0 \;\middle|\; \frac{1}{|f|^{2t}} \in L^1_x \right\}$

Show that $C_x(f) \in \mathbb{Q}$ (indeed, $C_x(f) = \operatorname{lct}(Z_f; x)$).

#

Positive line bdls

(L, h_0) a hol. line bdl with a s.m. metric h_0. $\rightsquigarrow$ Chern curvature form $\Theta = \frac{i}{2\pi}\Theta_{L,h_0} \in c_1(L)$

$\downarrow$

$(X_{/\mathbb{C}}, \omega)$ compact complex mfd with a Hermitian metric ω

Recall ananalytic def. of several positivity:

L is called nef if $\forall\, \varepsilon > 0$, $\exists$ a s.m. metric h_ε, s.t. $\frac{i}{2\pi}\Theta_{L,h_\varepsilon} > -\varepsilon\omega$.

$\iff \forall\, \varepsilon > 0$, $\exists\, \varphi_\varepsilon \in C^\infty(X, \mathbb{R})$ s.t. $\Theta + \frac{i}{\pi}\partial\bar\partial\varphi_\varepsilon > -\varepsilon\omega$

ample if $\exists$ a s.m. Hermitian metric $\tilde h$ s.t. $\frac{i}{2\pi}\Theta_{L,\tilde h} \geq \delta\omega$ for some $\delta > 0$

$\iff \exists\, \delta > 0$ and $\varphi \in C^\infty(X, \mathbb{R})$, s.t. $\Theta + \frac{i}{\pi}\partial\bar\partial\varphi \geq \delta\omega$

psef if $\exists$ a sing. metric $\tilde h$ s.t. $\frac{i}{2\pi}\Theta_{L,\tilde h} \geq 0$ in the sense of currents

$\iff \exists\, \psi \in L^1(X, \mathbb{R})$, s.t. $\Theta + \frac{i}{\pi}\partial\bar\partial\psi \geq 0$

big if $\exists$ a sing. metric $\tilde h$ s.t. $\frac{i}{2\pi}\Theta_{L,\tilde h} \geq \delta\omega$ in the sense of currents

$\iff \exists\, \delta > 0$, $\psi \in L^1(X, \mathbb{R})$ s.t. $\Theta + \frac{i}{\pi}\partial\bar\partial\psi \geq \delta\omega$.

Exercise 3. (true for any compact complex manifold, for simplicity you can assume X is Kaehler.)

Exer.: In the above analytic def., nef $\Rightarrow$ psef. and find an example s.t. psef $\not\Rightarrow$ nef.

algebro-geometric notion of positivity.

In this setting, assume that $X_{/\mathbb{C}}$ is projective.

L is called nef if $\forall$ irred. curve $C \subseteq X$, $L\cdot C = c_1(L)\cdot C \geq 0$

ample if for some $m \in \mathbb{N}$, mL is very ample.

psef if $a(L) = \lim_{k\to\infty} \{[D_k]\}$, where D_k is effective.

big if $k(L) = n$ (i.e. for $m \gg 1$, $h^0(X, mL) \sim O(m^n)$).

$\left(k(L) = \max \left\{ \operatorname{rank} \Phi_{|mL|} \;\middle|\; \Phi_{|mL|} : X \setminus B_s|mL| \longrightarrow \mathbb{P}(H^0(X, mL)) \text{ Kodaira map} \right\} \right)$.

Equivalence of analytic pos. and algebro-geometric pos..

In some arguments we need the following Lemma.

Lem. Assume L admits a sing. metric h ($\overset{\text{loc.}}{=} e^{-2\varphi}$) s.t. $\Theta(L, h) \geq \delta\omega$ for some $\delta > 0$.

If φ has the property that $\nu(\varphi, x) \geq n+s$ and $x \in E_1(\varphi)$ is isolated in $E_1(\varphi)$, then

$$H^0(X, K_X + L) \twoheadrightarrow J^s_x(K_X + L) \quad \text{— the } s\text{-jet at } x.$$

proof of the Lem:

Let P be a hol. polynomial of $\deg \leq s$ in V where V is a nbhd. of x.

χ a cut-off function on V s.t. $\operatorname{spt} \chi \subset\subset V$, $\chi \equiv 1$ near x.

e a local hol. frame of $K_X + L$ on V.

Let $g = \bar\partial (P \cdot \chi \otimes e) = P\, \bar\partial \chi \otimes e$, then g is a ~~$\bar\partial$-closed~~ $\bar\partial$-closed $(n,1)$ form with values in L, satisfying:

$$\int_X |g|^2 e^{-2\varphi} = \int_{\operatorname{spt}\chi} |g|^2 e^{-2\varphi} < \infty$$

$\Uparrow$

Since $g = 0$ near x, and $e^{-2\varphi} \in L^1_{loc}$ on $V \setminus \{x\}$

($x \in E_1(\varphi)$ is isolated in $E_1(\varphi)$) $\Downarrow$

By the L^2-existence thm, $\exists$ a $(n,0)$-form f with values in L s.t.

$$\bar\partial f = g \quad \text{and}$$

$$\int_X |f|^2 e^{-2\varphi} \le C \int_X |g|^2 e^{-2\varphi} < \infty$$

$$\not\ni\ \nu(\varphi, x) \ge n+s \overset{\text{def.}}{\Longrightarrow} e^{-2\varphi(z)} \gtrsim |z-x|^{-2(n+s)}$$

$\Bigg| \Longrightarrow \operatorname{ord}_x(f) > s$

Let $H = \chi P \otimes e - f$, then $\bar\partial H = 0$, i.e. $H \in H^0(X, K_X + L)$ and $J^s H = P$.

#

Equivalence:

① ampleness. Kodaira's embedding thm.

② psef.

· Assume L is psef in algebro-geometric sense, then $c_1(L) = \lim_{k\to\infty} \{D_k\}$, where D_k is effective divisor.

Consider the sequence of currents $[D_k]$, then the mass of $[D_k]$:

$$\|[D_k]\| = \int_X [D_k] \wedge \omega^{n-1} = \{D_k\}\cdot\{\omega^{n-1}\} \longrightarrow c_1(L)\cdot\{\omega^{n-1}\}.$$

ω is a fixed Kähler metric

In particular, $\|[D_k]\| \le C$ for some uniform $C>0$. (indep. of k)

Banach-Alaoglu thm $\Big| \Rightarrow$

$\exists$ a subsequence $[D_{k_\ell}] \longrightarrow T \ge 0$ for some positive current T.

In particular, $G(L)$ admits a positive $(1,1)$ current T

$\Leftrightarrow$ L has a sing. metric $\tilde{h}$ s.t. $G(L,\tilde{h}) \geq 0$ in the sense of currents.

- Assume L is psef. in the analytic sense., i.e. L has a sing. metric $h_L \overset{loc}{=} e^{-2\varphi_L}$ s.t.

$$\frac{i}{2\pi}\Theta_{h_L} = \frac{i}{\pi}\partial\bar\partial\varphi_L \geq 0, \quad \varphi_L \in L^1_{loc}$$

Take a point $x_0 \in X$ s.t. $\nu(\varphi_L, x_0) = 0$.

Let $\psi_0 = \chi\, n \log|z - x_0|$, where ~~spt~~ χ has compact spt and $\chi \equiv 1$ near x.

In particular, ψ_0 is s.m. on $X \setminus \{x_0\}$ and equals $n \log|z - x_0|$ near x_0.

Let A be an ample line bdl with a s.m. metric $h_A \overset{loc}{=} e^{-2\varphi_A}$, $\frac{i}{\pi}\partial\bar\partial\varphi_A > \delta\omega$.

For $m_0 \gg 1$, $\quad m_0\, G(A, h_A) + \frac{i}{\pi}\partial\bar\partial\psi_0 > \omega$.

We endow the line bdl $kL + m_0 A$ with the metric $h_L^k \cdot h_A^{m_0} \cdot e^{-2\psi_0} \overset{loc}{:=} e^{-2\varphi_k}$

then $\varphi_k = k\varphi_L + m_0\varphi_A + \psi_0$

$$\frac{i}{\pi}\partial\bar\partial\varphi_k = k\frac{i}{\pi}\partial\bar\partial\varphi_L + m_0\frac{i}{\pi}\partial\bar\partial\varphi_A + \frac{i}{\pi}\partial\bar\partial\psi_0 > \omega$$

$$\nu(\varphi_k, x_0) = \nu(\psi_0, x_0) = n$$

$\nu(\varphi_k, x) = k\,\nu(\varphi_L, x)$ for $x \in V_k \setminus \{x_0\}$, V_k a nbhd of x_0

< 1 for suitable V_k and any $x \in V_k \setminus \{x_0\}$, since ~~$\nu(\varphi_L, x_0)$~~ $\nu(\varphi_L, x_0) = 0$.

(Here we use the u.s.c. of $x \mapsto \nu(T, x)$)

Applying the Lem, $K_X + kL + m_0 A$ admits a non-zero section s_k for any $k \geq 1$.

Denote $D_k = s_k^{-1}(0)$, then:

$$G(L) = \frac{1}{k}\left\{[D_k] - m_0 G(A) - G(K_X)\right\} = \lim_k \left\{\left[\frac{1}{k} D_k\right]\right\}$$

Thus, L is psef in the algebro-geometric sense.

③ Bigness.

· Assume $k(L) = n$, i.e. $h^0(kL) \sim O(k^n)$ for $k \gg 1$.

Consider the exact sequence $0 \to \mathcal{O}(kL - A) \xrightarrow{s_A} \mathcal{O}(kL) \to \mathcal{O}_A(kL) \to 0$,

where A is a hypersurface s.t. $\mathcal{O}(A)$ ample.

$$\rightsquigarrow \quad 0 \to H^0(X, kL-A) \to \underset{O(k^n)}{H^0(X, kL)} \to \underset{\leq O(k^{n-1})}{H^0(A, kL|_A)} \to \cdots$$

$\Rightarrow$ for $k \geq k_0$, $H^0(X, kL-A) \neq \{0\}$, thus $kL = A + D$ for some $D \geq 0$

Then $\varphi_L := \frac{1}{k}(\varphi_A + \varphi_D)$ is a sing. metric s.t. $\frac{i}{\pi}\partial\bar\partial \varphi_L \geq \delta\omega$.

Exercise 4. prove that analytic bigness implies geometric bigness.

· For the other direction, Exer.

Hint: apply the Lem. to $kL = K_X + (kL - K_X)$ and endow $kL - K_X$ suitable metric s.t. kL generates 1-jets at generic points.

#

④ nefness.

- analytic nef $\Rightarrow$ alg. nef. , clear, since $c_1(L)\cdot C = \int_C c_1(L)$

- if $L\cdot C \geq 0$ for any irred. curve C, then $kL + A$ is ample for any $k \geq 1$ and some fixed ample line bdl A.

Write $L = \frac{1}{k}(kL+A) - \frac{1}{k}A$, then L has a s.m. metric with curvature $\geq -\frac{1}{k}c_1(A, h_A)$

#

Rmk. The positivity for line bdls can be generalized to any $\alpha \in H^{1,1}_{BC}(X, \mathbb{R})$ where X is a compact complex mfd.

Nadel vanishing theorem.

Thm. (X, ω) a Kähler mfd, X weakly pseudo-convex and X contains a Stein Zariski open set.

(E.g. X proj mfd)

L a line bdl on X with a sing. metric h s.t. the curvature current $c_1(L, h) \geq \delta\omega$.

Then $H^q(X, \mathcal{O}(K_X+L)\otimes \mathcal{J}(h)) = 0$ for any $q \geq 1$.

proof: $\forall\ q \geq 0$, $q \in \mathbb{N}$, let A^q = the sheaf of germs u of measurable sections of $\Lambda^{n,q} \otimes L$ s.t. $|u|^2 e^{-2\varphi}, |\bar\partial u|^2 e^{-2\varphi} \in L^1_{loc}$

Then we have a complex of sheaves:

$$(*)\quad 0 \to \mathcal{O}(K_X+L)\otimes \mathcal{J}(h) \to A^0 \xrightarrow{\bar\partial} A^1 \xrightarrow{\bar\partial} \cdots \to A^n \to 0$$

By the L^2-existence thrm, $(*)$ is exact.

every A^q is a sheaf of C^∞-module $\Rightarrow H^q(X, \mathcal{O}(K_X+L)\otimes\mathcal{J}(h)) \cong H^q(\Gamma(A^\cdot), \bar\partial)$

By the L^2-existence thrm again, $H^q(\Gamma(A^\cdot), \bar\partial) = 0 \quad \forall\, q \geq 1.$

(with some consideration for $L^2_{loc}(X)$ and $L^2(X)$)

#

Cor. (Kawamata-Viewheg vanishing)

$X/\mathbb{C}$: proj. mfd, F : a line bdl satisfying $mF = L + D$, L nef & big, D effective,

then $H^q(X, \mathcal{O}(K_X+F)\otimes \mathcal{J}(m^{-1}D)) = 0, \ \forall\, q \geq 1.$

Exercise 5. prove K-V vanishing thrm by using Nadel vanishing.

proof: Exer.

Hint: endow F with a sing. metric φ_F, s.t. $\frac{i}{\pi}\partial\bar\partial\varphi_F \geq \delta\omega$.

$\mathcal{J}(\varphi_F) = \mathcal{J}(m^{-1}D)$.

#