Motivation:

- X complex compact Kähler mfd. Ω_X^1 cotangent bundle, $K_X := \det \Omega_X^1$

positivity Ω_X^1 and K_X:

- → alg geom: $\boxed{\mathrm{Sym}^m \Omega_X^1}$ $\boxed{K_X^{\otimes m}}$ global sections. $\Downarrow$ metric ≥ 0, $\Uparrow$
- → analytic geom: $\exists\, h$ metric on Ω_X^1, K_X s.t. $i\Theta_h(\Omega_X^1) \geq 0$, $i\Theta_h(K_X) \geq 0$

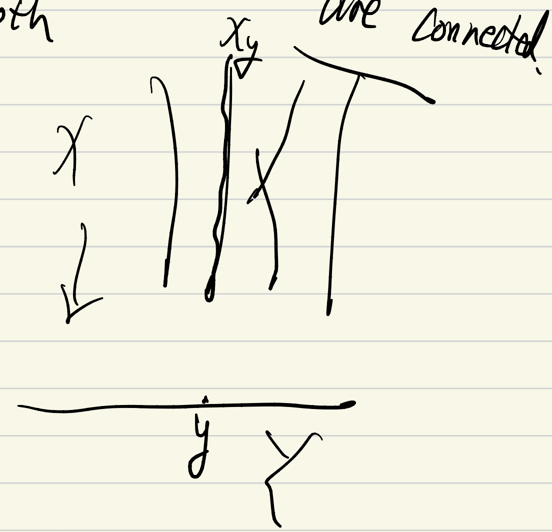
  - Kodaira
  - L^2-estimates
  - Ohsawa-Takegoshi extension

- We consider a family of Kähler mfds.

$\boxed{\pi : X \to Y}$ be a proper holo fibration between two Kähler mfds.

↑ surjective, fibers are connected

Rk: X smooth $\Rightarrow$ $X_y := \pi^{-1}(y)$ is smooth, $y \in Y$ generic pt



In analogy to absolute K_X we consider $K_{X/Y} := K_X \otimes \pi^*(K_Y^*)$.

In many cases, $\pi_*(mK_{X/Y}) \neq 0$, $m \in \mathbb{N}^*$ $\Rightarrow$

- $K_{X/Y}$ psef
- $(\pi_*(mK_{X/Y}), h) \succeq 0$.
- $R^k \pi_*(\Omega^{n-k}_{X/Y} \otimes L)$ curvature formula

$\rightsquigarrow$ ~~$\exists$~~ geom property of Y.

II. Basic ~~notes~~ properties in complex geom

(cf. Demailly: analytic methods in AG)

(1) Chern connection, curvature.

X complex mfd. $\underline{E} \to X$ holo vector bundle, h C^∞ herm metric on E.

Let $s \in C^\infty(X, \Omega^{p,q}_X \otimes E) \xrightarrow{\bar\partial} C^\infty(X, \Omega^{p,q+1}_X \otimes E)$.

$t \in C^\infty(X, \Omega^{p',q'}_X \otimes E)$

Fix a holo basis $\{e_1, \cdots, e_r\}$ of E

$\Rightarrow s = \sum s_i \otimes e_i$, s_i: (p,q)-form

$t = \sum t_i \otimes e_i$, t_i (p',q')-form.

Def: $\{s, t\} := \sum_{i,j} s_i \wedge \overline{t_j} \langle e_i, e_j \rangle_h \in C^\infty(X, \Omega_X^{p+q', q+p'})$

$\langle e_i, e_j \rangle_h \in \mathbb{C}$

- $\bar\partial s := \sum_i (\bar\partial s_i) \otimes e_i$ well defined.

Prop-Def: $\exists !$ (1,0)-connection:

$$\underline{\partial_h} : C^\infty(X, \Omega_X^{p,q} \otimes E) \to C^\infty(X, \Omega_X^{p+1,q} \otimes E)$$

s.t. $\forall s, t$

$$\partial\{s,t\} = \{\partial_h s, t\} + (-1)^{\deg s}\{s, \bar\partial t\}.$$

- $D_h := \partial_h + \bar\partial : C^\infty(X, \Omega_X^{\bullet} \otimes E) \to C^\infty(X, \Omega_X^{\bullet+1} \otimes E)$

$\uparrow$ Chern connection.

- $\underline{\sqrt{-1}\Theta_h} := \sqrt{-1}\, \underline{D_h \circ D_h} \in \underline{C^\infty(X, \Omega_X^{1,1} \otimes \mathrm{End}(E))}$.

Exo: $d\circ d = 0 \Rightarrow \{\sqrt{-1}\Theta_h(u), v\} = \{u, \sqrt{-1}\Theta_h(v)\}$ $\boxed{(*)}$

$\forall\, u, v$ (p,q) with value in E.

Cor 1: $\mathrm{rk} E = 1$: $\mathrm{End}(E) =$ trivial line on X.

$\Rightarrow \sqrt{-1}\Theta_h$ is a $(1,1)$-form.

$(*) \Rightarrow \sqrt{-1}\Theta_h$ is a real $(1,1)$-form.

$\boxed{\text{Cor 2}}$: $\mathrm{rk} E \geq 1$: $\underline{\sqrt{-1}\Theta_h(E)}$ induce a $\underline{\text{herm form}}$ on $\boxed{T_X \otimes E}$

proof: Fix a holo coord $\underline{(z_1, \dots, z_n)}$ of X.

$$\sqrt{-1}\Theta_h = \sum \Theta_{ij}\, dz_i \wedge d\bar z_j \quad \text{where } \Theta_{ij} \in \mathrm{End}(E).$$

Let $\alpha = \sum_i \frac{\partial}{\partial z_i} \otimes \boxed{\alpha_i} \in \underline{T_X \otimes E}$, $\quad \underline{\alpha_i \in E}$

$$\langle i\Theta_h(E)\alpha, \alpha\rangle := \sum_{i,j} \langle \underbrace{\underbrace{\Theta_{ij}(\alpha_i)}_{\in E}}_{}, \underbrace{\alpha_j}_{\in E} \rangle_h \in \mathbb{C}$$

$(*) \Rightarrow \langle i\Theta_h(E)\alpha, \alpha\rangle \in \mathbb{R}$

<u>Def</u>: We say $\sqrt{-1}\Theta_h(E)$ is Nakano semipositive (positive)

if $\langle i\Theta_h(E)\alpha, \alpha\rangle \geq 0 \quad \forall \alpha \in T_X \otimes E$

$(> 0 \quad \forall \alpha \neq 0 \in T_X \otimes E)$

• $\sqrt{-1}\Theta_h(E)$ is Griffith semipositive / positive

if $\langle i\Theta_h(E)\alpha, \alpha\rangle \geq 0 \quad \forall \alpha = \frac{\partial}{\partial z_i} \otimes \alpha_i \in \underline{T_X \otimes E}$

> 0

$\alpha_i \in E$

<u>Exo</u>: Let $u \in C^\infty(X, E)$

$\underline{\sqrt{-1}\Theta_h(E) \geq 0}$ Griffith $\Leftrightarrow$ $\underline{\{\sqrt{-1}\Theta_h(E)(u), u\}} \geq 0$

(1,1)-form.

<u>Rk</u>: Nakano $\Rightarrow$ Griffith

$\not\Leftarrow$

• Griffith ≥ 0 appears in many situation in alg geom.

For example, $T_X(-mK_{X/Y}) \geq 0$ "Griffith"

• <u>Nakano</u>: <u>vash.y thm, O-T</u>

line bundle

Exo: $(\underline{E}, \underline{h}) \rightsquigarrow (\underline{\det E}, \det h)$

· $i\Theta_{\det h}(\det E) := \mathrm{Tr}(i\Theta_h(E))$.

$\boxed{\text{Prop}}$: X complex mfd. Let S, E, Q three complex hol vector bundle on X, and exact sequence.

$$0 \to S \xrightarrow{\pi_1} E \xrightarrow{\pi_2} Q \to 0$$

Fix h_E be a herm metric on E.

Then h_E induces a herm metric $\boxed{h_S \text{ on } S}$.

$x \in X$ $\quad$ " — — — — h_Q on Q.

We have: (1) $\forall u \in S_x$

· $\{i\Theta_{h_S}(u), u\} \leq \{i\Theta_{h_E}(\pi_1(u)), \pi_1(u)\}$

(2) $\forall v \in Q$, let $\tilde{v} \in E$ s.t $\begin{cases} \pi_2(\tilde{v}) = v \\ \|\tilde{v}\|_{h_E} \text{ is min.} \end{cases}$

$$\{i\Theta_{h_Q}(v); v\} \geq \{i\Theta_{h_E}(\tilde{v}), \tilde{v}\}$$

In particular: (E, h_E) if $i\Theta_{h_E}(E) \geq 0 \Rightarrow i\Theta_{h_Q}(Q) \geq 0$.

Exo: $i\Theta_{h_E}(E) \geq 0$ Griffith. We consider $\mathbb{P}(E)$

$\Rightarrow i\Theta_{h_1}(\mathcal{O}_{\mathbb{P}(E)}(1)) \geq 0$

Indicatu:

$$\begin{tikzcd} \mathcal{O}(1)_{\mathbb{P}(E)} \arrow[d] & E \arrow[d] \\ \mathbb{P}(E) \arrow[r, "\pi"] & X \end{tikzcd}$$

$$\begin{tikzcd} \pi^* E \arrow[rr, "\text{surjective}"] \arrow[dr] & & \mathcal{O}(1)_{\mathbb{P}(E)} \arrow[dl] \\ & \mathbb{P}(E) & \end{tikzcd}$$

$i\Theta_h(E) \geq 0$

$\Rightarrow i\Theta_{\pi^* h}(\pi^* E) \geq 0$.

(2) primitive form and Hodge theory:

On $\mathbb{C}$: $\quad i\,dz \wedge d\bar{z} = 2\,dx \wedge dy > 0$ volume form

On $\mathbb{C}^n$: $\quad dz := dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n$

$$\boxed{i^{n^2} dz \wedge d\bar{z}} = 2^n dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge \cdots$$

> 0 volume form.

Cor. X^n complex mfd. α $(n,0)$-form $\Rightarrow \alpha = f(z) \cdot dz$

$\Rightarrow i^{n^2} \{\alpha, \alpha\} \geq 0$ volume form.

Note. If $\boxed{1 \leq p \leq n-1}$, $\nexists$ $c \neq 0$ constant s.t.

$$c \cdot \{\alpha, \alpha\} \geq 0 \quad \forall\ \alpha \ (p, n-p)\text{-form}.$$

Prop-Def: X^n complex mfd. ω_X herm metric on X.

α is $(p, n-p)$-form

We say: $\underline{\alpha}$ is $\underline{\omega_X\text{-primitive}}$, if

$$\underbrace{\omega_X \wedge \alpha}_{(p+1,\, n-p+1)\text{-form}} \equiv 0$$

- $\exists\ \underset{\text{constant}}{C_{p,n}} \neq 0$ s.t. $\underline{C_{p,n} \cdot \{\alpha, \alpha\} \geq 0}$ $\forall\ \underset{(p,\, n-p)\text{-form}}{\alpha}$ ω_X-primitive

- Hodge theory and Dolbeault cohomology:

$(X^n, \underline{\omega_X})$ complex mfd., ω_X herm metric.

$E \to X$ holo vector bundle. $\underline{h_E}$ C^∞ herm metric.

- $\bar\partial : C^\infty(X, \Omega_X^{p,q} \otimes E) \longrightarrow C^\infty(X, \Omega_X^{p,q+1} \otimes E)$
- We can define $\langle\,,\,\rangle$ on $C^\infty(X, \underline{\Omega_X^{p,q}} \otimes E)$. ($S \in$)

$$\boxed{\langle S, S \rangle} := \int_X |S|^2_{\omega_X, h_E} \cdot \omega_X^n \geq 0.$$

- **Prop.**: $\exists\ \underline{\bar\partial^*} : C^\infty(X, \Omega_X^{p,q} \otimes E) \to C^\infty(X, \Omega_X^{p,q-1} \otimes E)$

s.t. $\forall$ S $(p,q) \otimes E$, $\underline{t}$ $(p, q+1) \otimes E$

$$\langle \bar\partial S, t \rangle = \langle S, \bar\partial^* t \rangle$$

- Def (Dolbeault): X compact complex mfd.

$$\boxed{H^{p,q}(X, E)} := \frac{\operatorname{Ker} \bar\partial : C^\infty(X, \Omega_X^{p,q} \otimes E) \to C^\infty(X, \Omega_X^{p,q+1} \otimes E)}{\operatorname{Im} \bar\partial : (C^\infty \cdots \Omega_X^{p,q-1} \otimes E) \to C^\infty(X, \Omega^{p,q}_X \otimes E)}$$

Thm (Hodge decomp): X compact. $\Delta'' = [\bar\partial, \bar\partial^*]$

- $C^\infty(X, \Omega_X^{p,q} \otimes E) = \boxed{\ker \Delta''_{\omega_X, E} \oplus \operatorname{Im} \bar\partial} \oplus \operatorname{Im} \bar\partial^*$

$\langle , \rangle$ $\langle , \rangle$

- $\ker \Delta''_{\omega_X, E} \simeq H^{p,q}(X, E)$.

RK: construct holo section

$\Uparrow$

$\boxed{\text{solve } \bar\partial\text{-equation}}$

If α (p,q)-$\bar\partial$-closed $\Rightarrow \alpha \in \ker \Delta'' \oplus \operatorname{Im} \bar\partial$

If we can prove $\alpha \perp \ker \Delta''$

$\Rightarrow \alpha \in \operatorname{Im} \bar\partial$

$\Longleftrightarrow$ we can solve $\bar\partial$-equation.

(3) Ohsawa-Takegoshi: extension thm.

An important question in A-G: $\boxed{X \text{ proj mfld}}$, $Y \subset X$ subvariety.

$\boxed{L \to X}$ holo line bundle.

If $L|_Y$ has some positivity, i.e., $H^0(Y, L|_Y^{\otimes m}) \neq 0$ $m \in \mathbb{N}^*$.

Q: $H^0(X, L^{\otimes m}) = ?$

Local version: $\underline{X = \mathbb{A}^n} = \underbrace{\mathbb{D}^1}_{\text{unit disc}} \times \overbrace{\mathbb{D}^1 \times \mathbb{D}^1 \times \cdots \times \mathbb{D}^1}^{n \text{ times}} \subset \mathbb{C}^n$.

- $\underline{Y} = X \cap \underset{\substack{\uparrow \\ \text{hyperplane in } \mathbb{C}^n}}{H} \subset X$

- φ psh function on X

- $\boxed{f \in H^0(Y, \mathcal{O}_Y)}$ s.t $\boxed{\int_Y |f|^2 e^{-\varphi} dV_Y} < +\infty$.

Euclidean volume.

Then: $\exists F \in H^0(X, \mathcal{O}_X)$, s.t

- $\boxed{\int_X |F|^2 e^{-\varphi} dV_X} \leq C \cdot \int_Y |f|^2 e^{-\varphi} dV_Y$

- $F|_Y = f$.

where $\underline{C}$ depends only on $\underline{(X, Y)}$. (independent of f, φ)

RK: Applications: φ, ψ psh

Then. $\mathcal{I}(\varphi+\psi) \subset \mathcal{I}(\varphi) \cdot \mathcal{I}(\psi)$.

- φ psh $\overset{\text{Demailly}}{\Longrightarrow}$ $\varphi_k \searrow \varphi$

$$\boxed{\varphi_k = \ln \sum |f_i|^2 + C^\infty}$$

h-$\mathcal{L}^2$

Global version: Let $\pi: X \to Y$ be a proper fibration between two Kähler mfds. (more general, X weakly pseudo-convex)

$L \to X$ holo line bundle, $\Theta_{h_L}(L) \geq 0$

h_L (possible) singular.

Case 1: $Y = \boxed{\Delta^n}$ polydisc in $\mathbb{C}^n$

Let $0 \in Y$ be the central pt.

If $X_0 := \pi^{-1}(0)$ is smooth, then $\forall f \in H^0(X_0, K_{X_0} + L)$

with $\boxed{\int_{X_0} |f|^2_{h_L} < +\infty}$ $\leftarrow$ $|f|^2_{h_L}$ $\leftarrow$ volume form

we can find a $F \in H^0(X, K_X + L)$, s.t.

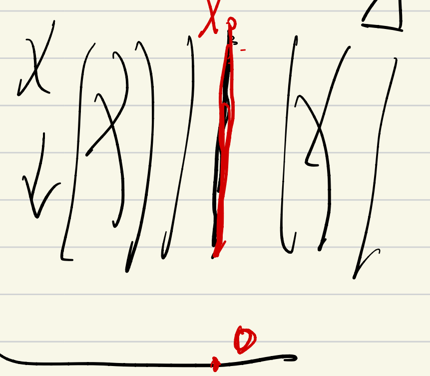
$$\begin{cases} \int_X |F|^2_{h_L} \leq \boxed{C} \cdot \int_{X_0} |f|^2_{h_L} \\ F|_{X_0} = f \wedge \pi^* dz \end{cases}$$

$f = g\, dz \otimes e_L$

$|f|^2_{h_L} = i^{n^2} |g|^2 \, dz \wedge d\bar{z} \cdot |e_L|^2_{h_L}$

$dz = dz_1 \wedge \cdots \wedge dz_n$ ≥ 0

on Δ^n



Rk: C is indepent of f, h_L

In fact: Guan-Zhou:

$$\boxed{C = \mathrm{vol}(\Delta^n)}$$

$\boxed{X = Y = \Delta^n}$, π is id, (L, h_L) trivial. $Y = \Delta^n$

$\mathrm{vol}(\Delta^n) \cdot |F(0)|^2 \leq \boxed{\int_{\Delta^n} |F|^2 \leq \mathrm{vol}(\Delta^n) \cdot |F|^2(0)}$

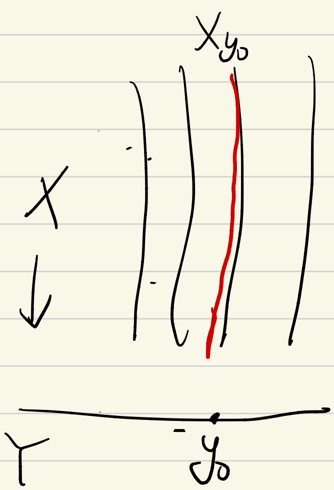
(2) · Y proj mfd (quasi-proj). $L \to X$ holo line bdl, $i\Theta_{h_L}(L) \geq 0$.

· Let $y_0 \in Y$, s.t $X_{y_0} = \pi^{-1}(y_0)$ is smooth

Let U be a nb of $y_0 \in Y$.

and $(z_1, \dots, z_n)$ be a holo coord

$\exists$ A_Y ample line bundle on Y, h_Y C^∞ metric on A_Y,



s.t $\forall f \in H(X_{y_0}, K_{X_{y_0}} + L + \pi^* A_Y)$

with $\boxed{\int_{X_{y_0}} |f|^2_{h_L, h_A} < +\infty}$

$\exists$ $F \in H^0(X, K_X + L + \pi^* A_Y)$

s.t $\left\{ \begin{array}{l} \boxed{\int_X |F|^2_{h_L, h_A}} \leq C \cdot \boxed{\int_{X_{y_0}} |f|^2_{h_L, h_A}} \\ F|_{X_0} = f \wedge \pi^* dz. \end{array} \right.$

Here C is independent of f, h_L.

A_Y depends only on Y (independent of $L, f \dots$) $\square$

Next lecture:

$\pi : X \longrightarrow Y$ proper fibration between two Kähler mfds

$L \to X$ h_L $i\Theta_{h_L}(L) \geq 0$.

If $\pi_*(K_{X/Y} \otimes L \otimes \mathcal{I}(h_L)) \neq 0$

$\Rightarrow$ · $K_{X/Y} \otimes L$ is psef

· $(\pi_*(K_{X/Y} \otimes L \otimes \mathcal{I}(h_L)), h) \succeq 0$ Griffith

· $\pi_*(mK_{X/Y} \otimes L \cdots) \succeq 0$.