Lecture 2 2022.8.9

§5 nefness of $f_*\omega_{X/Y}^{\otimes m}$

Today's first goal is to establish:

Th 5.1 (Thm 4.13) $\quad$ f: projective

$f: X \to Y$: smooth morphism with connected fibers /$\mathbb{C}$

X, Y: smooth proj varieties

$\Rightarrow f_*\omega_{X/Y}^{\otimes m}$: nef locally free sheaf for $^\forall m \geq 1$.

Yesterday, I proved $f_*\omega_{X/Y}$: nef locally free sheaf

Th 5.2 (Siu, Invariance of plurigenera)

$\underline{\underline{f}} : X \to Y$: smooth projective morphism with connected fibers

X, Y : smooth varieties

$$\Rightarrow \dim_{\mathbb{C}} H^0(X_y, \overset{\overset{P_m(X_y)}{\shortparallel}}{\mathcal{O}_{X_y}(mK_{X_y})})$$

is independent of $y \in Y$

for $\forall m \geq 1$, where $\underset{\text{smooth proj var}}{X_y} := f^{-1}(y)$.

Th 5.2 can be proved as an easy application of $\boxed{\text{Ohsawa-Takegoshi's } L^2 \text{ extension}}$ theorem (Păun's proof).

when $\chi(X_y) = \dim X_y \Rightarrow \exists$ algebraic proof of Th 5.2 (Siu) ③

Rem 5.3 The proof of Th 5.2 is complex analytic. There are no algebraic proofs of Th 5.2.

As an obvious corollary, we have:

Cor 5.4 $f: X \to Y$: smooth projective morphism with connected fibers

X, Y: smooth varieties

$\Rightarrow$ $f_* \omega_{X/Y}^{\otimes m}$: locally free for $\forall m \geq 1$

flat base change theorem
+ Th 5.2

By Cor 5.4, all we have to do is to prove the nefness of $f_*\omega_{X/Y}^{\otimes m}$ for Thm 5.1.

Yesterday

Hodge theory $\longrightarrow$ loc. freeness of $f_*\omega_{X/Y}$

Kollár vani + CM regularity $\longrightarrow$ nefness of $f_*\omega_{X/Y}$

Today

Invariance of plurigenera $\longrightarrow$ loc. freeness of $f_*\omega_{X/Y}^{\otimes m}$

Popa-Schnell's freeness $\longrightarrow$ nefness of $f_*\omega_{X/Y}^{\otimes m}$

a generalization of

Th 5.5 (Kollár's vanishing theorem)

X: smooth proj var $/\mathbb{C}$

Y: proj var

$f : X \to Y$: surjective morphism

Δ: effective $\mathbb{Q}$-divisor on X s.t

$\operatorname{Supp}\Delta$: s.n.c divisor,

simple normal crossing

the coefficients of Δ are in $[0, 1)$.

H: ample Cartier div on Y

L: Cartier div on X

Assume $L - (K_X + \Delta) \sim_{\mathbb{Q}} f^*H$

$$\Rightarrow H^i(Y, R^j f_* \mathcal{O}_X(L)) = 0 \text{ for } \forall i > 0, \ \forall j \geq 0.$$

<u>Rem</u> 5.6 In Thm 5.5, we put $\Delta = 0$,

$$L = K_X + f^*H. \Rightarrow L - K_X = f^*H$$

Then Thm 5.5 says that

$$H^i(Y, \boxed{R^j f_* \mathcal{O}_X(K_X)} \otimes \mathcal{O}_Y(H)) = 0$$

for $^\forall i > 0$, $^\forall j \geq 0$

This is Kollár's original vanishing theorem (Thm 4.5).

$$R^j f_* \mathcal{O}_X(L)$$

$$\simeq R^j f_* \mathcal{O}_X(K_X) \otimes \mathcal{O}_Y(H)$$

projection formula

Thm 5.7 X: smooth proj var $/\mathbb{C}$

$\mathcal{L}$: semi-ample line bundle on X

$s \in |\mathcal{L}^{\otimes k}|$ for some $k \in \mathbb{Z}_{>0}$

$$\Rightarrow \quad \otimes s : H^i(X, \omega_X \otimes \mathcal{L}^{\otimes \ell}) \longrightarrow H^i(X, \omega_X \otimes \mathcal{L}^{\otimes (k+\ell)})$$

is injective for $^\forall i$ and $^\forall \ell \in \mathbb{Z}_{>0}$.

Thm 5.7 is Kollár's injectivity theorem.

$\mathcal{L}$: semi-ample $\overset{\text{def}}{\Longleftrightarrow}$ $|\mathcal{L}^{\otimes m}|$: free for some $m \in \mathbb{Z}_{>0}$

Thm 5.7 is also a generalization of Kodaira's vanishing.

In Th 5.7, we further assume $\mathcal{L}$ is ample.

$$H^i(X, \omega \otimes \mathcal{L}) \hookrightarrow \underbrace{H^i(X, \omega \otimes \mathcal{L}^{\otimes(k+1)})}_{\substack{\parallel \\ 0}}$$

for some $k \gg 0$.

If $i > 0$, then the RHS $= 0$ by Serre's vanishing theorem.

$\Longrightarrow$ $H^i(X, \omega \otimes \mathcal{L}) = 0$ for $\forall i > 0$.

We can recover Kodaira's vanishing.

<u>Thm 5.8</u> (Kollár's vanishing thm and torsion-freeness)

X: smooth proj var $/\mathbb{C}$

$f: X \to Y$: surjective morphism

Y: proj var

$\Rightarrow$ ✓ ① (Torsion-freeness)

$R^i f_* \omega_X$: torsion-free coherent sheaf for $\forall i$

◎ ✓ ② (Vanishing theorem, Thm 4.5)

$$H^i(Y, R^j f_* \omega_X \otimes \mathcal{A}) = 0$$

for $\forall i > 0$, $\forall j$,

where $\mathcal{A}$ is an <u>ample</u> invertible sheaf on Y.

Kollár's injectivity

Rem 5.9 ① Thm 5.7 $\Longleftrightarrow$ Thm 5.8
equivalent

② We can prove Thm 5.7 as an easy application of Hodge theory.

③ We can also prove Thm 5.7 by using the theory of harmonic forms.

(Bochner's trick due to Enoki).

Thm 5.7 is not difficult to prove.

Thm 5.7 $\Longrightarrow$ Thm 5.8

routine work.

Rem 5.10

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\boxed{\text{Kodaira vanishing}} \arrow[r, "\mathbb{Q}\text{-div gen}"] \arrow[d, "\text{gen}"] & \text{Kawamata-Viehweg vanishing thm} \arrow[d, "\text{gen}"] \\
\text{Kollár's vanishing theorem (Thm 4.5)} \arrow[r, "\mathbb{Q}\text{-div gen}"'] & \boxed{\text{Thm 5.5}}
\end{tikzcd}
$$

Thm 4.5 = Thm 5.8 ②

By the usual covering trick, we can prove Th 5.5 by Th 4.5.

$$
\begin{array}{ccc}
\text{Thm 4.5} & \Longrightarrow & \text{Thm 5.5} \\
 & \uparrow & \\
 & \text{covering trick} & \\
 & \downarrow & \\
\text{Kodaira vanishing} & \Longrightarrow & \text{Kawamata-Viehweg vanishing}
\end{array}
$$

Rem 5.11 Kodaira's vanishing and Kollár's vanishing follow from the E_1-degeneration of the Hodge-to-de Rham spectral sequence.

$$E_1^{p,q} = H^q(X, \Omega_X^p) \Rightarrow H^{p+q}(X, \mathbb{C}).$$

By using the theory of MIXED Hodge structures, we have generalizations.

This sequence degenerates at E_1.

$\parallel$

Hodge decomposition.

Mixed generalization

X: smooth proj var $/\mathbb{C}$

D: s.n.c div on X

$\checkmark$ $E_1^{p,q} = H^q(X, \Omega_X^p(\log D) \otimes \mathcal{O}_X(-D))$

$$\Rightarrow H_c^{p+q}(X \setminus D, \mathbb{C})$$

degenerates at E_1.

On Enoki's proof on Th 5.7

$$\underline{H^i(X, \omega_X \otimes \mathcal{L}^{\otimes \ell})}$$

$$\cong$$

$$\underline{\mathcal{H}^{n,i}(\mathcal{L}^{\otimes \ell})} \ni \underline{\varphi} \qquad n = \dim X$$

$= \{ \mathcal{L}^{\otimes \ell}$-valued harmonic (n,i)-forms $\}$

By Bochner's trick,
+ semipositivity of $\mathcal{L}$

$\Longrightarrow$ $\underline{\underline{s}} \otimes \varphi$ is also harmonic.

$\mathcal{H}^{n,i}(\mathcal{L}^{\otimes(k+\ell)})$

$\otimes s : \mathcal{H}^{n,i}(\mathcal{L}^{\otimes \ell}) \to \mathcal{H}^{n,i}(\mathcal{L}^{\otimes(k+\ell)})$,

is obviously injective.

homework

Exercise 5.12

① Check

Kollár's injectivity (Thm 5.7)

$\Longrightarrow$ Kollár's vanishing (Thm 5.8 ②)

spectral sequence, Serre's vanishing, induction on dim.

② Check

Kollár's injectivity

$\Longrightarrow$ Kollár's torsion-freenes (Thm 5.8 ①)

similar

③ Check

Kollár's vanishing + torsion-freeness

$\Longrightarrow$ Kollár's injectivity.

As a very clever application of Th5.5, we have:

Th 5.13 (Popa-Schnell)

$f : X \to Y$: surjective morphism

X : smooth proj var $/\mathbb{C}$

Y : proj var with $\boxed{\dim Y = n}$

$\boxed{k} \in \mathbb{Z}_{>0}$

$\mathcal{L}$: ample line bundle on Y s.t

$|\mathcal{L}|$: free

$\Rightarrow \boxed{H^i(Y, f_*\omega_X^{\otimes k} \otimes \mathcal{L}^{\otimes \ell}) = 0}$ ← similar to Kollár's vanishing

for $\forall i > 0$, $\forall \ell \geq nk + k - n$.

$\Rightarrow \boxed{f_*\omega_X^{\otimes k} \otimes \mathcal{L}^{\otimes \ell}}$: globally generated

for $\forall \ell \geq k(n+1)$.

$\qquad\qquad\qquad\qquad \| $

$\qquad\qquad (nk + k - n) + n$

↑

Prop 4.7

Castelnuovo-Mumford regularity

Proof of Th 5.13

We assume $f_*\omega_X^{\otimes k} \neq 0$.

We put

$$\mathcal{M} := \mathrm{Im}(f^*f_*\omega_X^{\otimes k} \to \omega_X^{\otimes k}).$$

By taking suitable blow-ups,

we may assume

- $\mathcal{M}$: invertible sheaf
- $\omega_X^{\otimes k} = \mathcal{M} \otimes \mathcal{O}_X(E)$
- $E \geq 0$, Supp E : s.n.c divisor

(This is a standard argument based on Hironaka's resolution theorem)

$$\mathcal{I} := \mathrm{Im}(f^*f_*\omega_X^{\otimes k} \to \omega_X^{\otimes k}) \otimes \omega_X^{\otimes(-k)}$$

$\mathcal{I}$ is an ideal sheaf on X

Then we apply principalization to $\mathcal{I}$.

Let $m \geq 0$ be the smallest nonnegative integer such that

$$\boxed{\underline{f_*\omega_X^{\otimes k}} \otimes \mathcal{L}^{\otimes m}} \checkmark$$

is generated by global sections.

By construction,

$$\begin{cases} \mathcal{M} = \omega_X^{\otimes k} \otimes \mathcal{O}_X(-E) \checkmark \\ \underline{f^*f_*\mathcal{M} \twoheadrightarrow \mathcal{M}} : \text{surjective} \checkmark \\ \underline{f_*\mathcal{M} = f_*\omega_X^{\otimes k}} \checkmark \end{cases}$$

Hence

$$\mathcal{M} \underline{\otimes f^*\mathcal{L}^{\otimes m}} = \omega_X^{\otimes k} \otimes \mathcal{O}_X(-E) \underline{\otimes f^*\mathcal{L}^{\otimes m}}$$

is generated by global sections.

$f_*\omega_X^{\otimes k} \otimes \mathcal{L}^{\otimes m}$: gen. by global sec.

$\Rightarrow \underline{f_*\mathcal{M} \otimes \mathcal{L}^{\otimes m}}$: " "

$\Rightarrow \mathcal{M} \otimes f^*\mathcal{L}^{\otimes m}$: "

We take a general member

free linear system

$$\underline{D} \in |\underline{\omega_X^{\otimes k} \otimes \mathcal{O}_X(-E) \otimes f^* \mathcal{L}^{\otimes m}}|$$

Since D: general, $\mathrm{Supp}(D+E)$: s.n.c div.

We write $\mathcal{L} = \mathcal{O}_Y(L)$. L: Cartier div.

Then $\underline{kK_X + mf^*L \sim D+E}$. $\times \frac{k-1}{k}$

$$\Rightarrow (k-1)K_X \sim_{\mathbb{Q}} \underline{\frac{k-1}{k}D + \frac{k-1}{k}E - \frac{k-1}{k}mf^*L}$$

obvious

obvious

$$\Rightarrow kK_X - \lfloor \frac{k-1}{k}E \rfloor + lf^*L$$

$$\sim_{\mathbb{Q}} \underline{K_X} + \boxed{\frac{k-1}{k}D + \left\{\frac{k-1}{k}E\right\}} \Delta$$

$$+ \underline{\left(l - \frac{k-1}{k}m\right)} f^*L.$$

We put $\Delta := \underline{\frac{k-1}{k}D} + \left\{\frac{k-1}{k}E\right\}$

$\Delta \in [0, 1)$ fractional

Rem 5.14 B : $\mathbb{Q}$-div

$\Rightarrow$ $\lfloor B \rfloor$: integral part of B

$\{B\} := B - \lfloor B \rfloor$: fractional part of B.

By construction, $\mathrm{Supp}\,\Delta$: s.n.c div.

the coefficients of $\Delta \in [0, 1)$.

Therefore, if $l - \frac{k-1}{k} m > 0$, then

$$H^i(Y, f_*\mathcal{O}_X(kK_X - \lfloor \tfrac{k-1}{k} E \rfloor) \otimes \mathcal{O}_Y(lL)) = 0 \quad \text{—} (*) \checkmark$$

Th5.5

for $\forall i > 0$.

$$kK_X - \lfloor \tfrac{k-1}{k} E \rfloor + lf^*L - (K_X + \overset{[0,1)}{\Delta}) \quad \underset{\text{s.n.c}}{\mathrm{Supp}\,\Delta}$$

$$\sim_{\mathbb{Q}} \underbrace{(l - \tfrac{k-1}{k} m)}_{\substack{\vee \\ 0}} \underset{\text{ample}}{f^*L}$$

By the definition of E,

$$\underline{f_*\omega_X^{\otimes k}} \overset{\text{by def of } E}{=} \underline{f_*\mathcal{O}_X(kK_X - E)}$$

$$\subseteq f_*\mathcal{O}_X\Big(kK_X - \underbrace{\llcorner \tfrac{k-1}{k} E \lrcorner}_{\geq 0}\Big)$$

$$\subseteq f_*\omega_X^{\otimes k}.$$

Thus, $\underline{f_*\mathcal{O}_X(kK_X - \llcorner \frac{k-1}{k} E \lrcorner) = f_*\omega_X^{\otimes k}}$

By ⊛, we have

$$H^i(Y, f_*\omega_X^{\otimes k} \otimes \mathcal{L}^{\otimes l}) = 0$$

for $^\forall i > 0$ and $\underline{l > \frac{k-1}{k} m}$.

By Prop 4.7 (Castelnuovo - Mumford regularity),

✓ $\underline{f_*\omega_X^{\otimes k} \otimes \mathcal{L}^{\otimes l}}$ is generated by global sections for $\underline{l > \frac{k-1}{k} m + n}$.

By the choice of m,

$$m \leqq \underbrace{\frac{k-1}{k} m + n + 1}$$

$$\therefore \quad m \leqq k(n+1)$$

Hence, if $\underbrace{l > \frac{k-1}{k} k(n+1)}$

$$= kn + k - n - 1,$$

$\Longleftrightarrow l \geqq kn + k - n$

then

$$H^i(Y, f_* \omega_X^{\otimes k} \otimes \mathcal{L}^{\otimes l}) = 0$$

for $\forall i > 0$.

This is what we wanted. //

[Popa-Schnell proved this theorem for log canonical pairs.

Proof of Thm 5.1

The proof of Thm 4.1 works!

We consider:

$$f^s : \underbrace{X \times_Y X \times_Y \cdots \times_Y X}_{s\text{-times}} \longrightarrow Y$$

f: smooth
X, Y: smooth

s-fold fiber product of $f : X \to Y$.

As in the proof of Th 4.1, we have:

✓ **Claim** $f^s_* \omega_{X/Y}^{\otimes m} \simeq \bigotimes^s f_* \omega_{X/Y}^{\otimes m}$

for $^\forall s \geq 1$, $^\forall m \geq 1$.

Note that we have already known that $\boxed{f_* \omega_{X/Y}^{\otimes m} \text{ is locally free}}$.

Yesterday, I proved Clai[m] when $m = 1$.

We take an ample invertible sheaf $\mathcal{L}$ on Y s.t $|\mathcal{L}|$: free.

We put $\underline{\underline{\mathcal{M}}} := \omega_Y^{\otimes m} \otimes \mathcal{L}^{\otimes m(\dim Y+1)}$

smooth smooth

By applying Thm ~~5.9~~ 5.13 to $f^s : X^s \to Y$,

Popa-Schnell smooth

$$\underline{f^s_* \omega_{X^s/Y}^{\otimes m} \otimes \mathcal{M}} = f^s_* \omega_{X^s}^{\otimes m} \otimes \mathcal{L}^{\otimes m(\dim Y+1)}$$

is generated by global sections for $^\forall s \geq 1$.

By Claim,

$$\underline{\Big(\overbrace{\bigotimes^s f_* \omega_{X/Y}^{\otimes m}}^{\mathcal{E}}\Big) \otimes \mathcal{M}}$$

$\mathcal{M}$ is independent of s.

is generated by global sections for $\underline{\underline{^\forall s \geq 1}}$

By Lem 4.11, we obtain that

$$f_* \omega_{X/Y}^{\otimes m}$$

is a nef locally free sheaf.

We finish the proof of Th 5.1.

$\mathcal{E}^{\otimes s} \otimes \mathcal{M}$ is gen. by global sections for $\forall s \geqq 1$

$\Longrightarrow \mathcal{E}$: nef.

<u>Rem 5.15</u> In Thm 5.1, we assume that $f: X \to Y$: smooth.
This assumption is very strong.
Hence Thm 5.1 is not so useful for geometric applications. //

$\Rightarrow$ We want to generalize Thm 5.1!

$\begin{cases} X, Y: \text{both proj} \\ f: X \to Y: \text{smooth} \end{cases}$

very special situation.

<u>Main Goal</u> $f: X \to Y$ weakly s.s morphism

$\overset{?}{\Rightarrow}$ $f_* \omega_{X/Y}^{\otimes m}$ is nef loc free sheaf

§6 weakly semistable morphisms

Def 6.1 (Weakly semistable morphisms)

set

$f: X \to Y$: weakly semistable

Zariski open   ← normal var

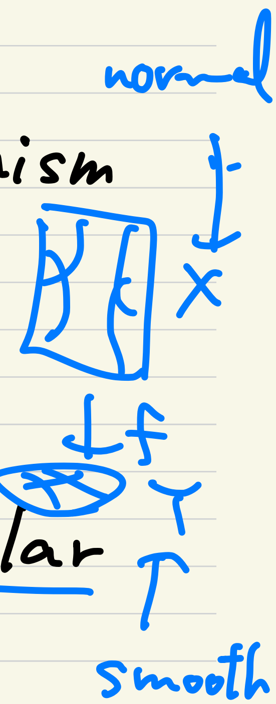
$\overset{\text{def}}{\Longleftrightarrow}$ (i) $(U_X \subset X)$, $(U_Y \subset Y)$: toroidal embeddings s.t. $U_X = f^{-1}(U_Y)$

(ii) $f: X \to Y$: toroidal with respect to $(U_X \subset X)$ and $(U_Y \subset Y)$

(iii) f : equidimensional

✓ (iv) all the fibers of the morphism f are reduced

(v) Y : smooth variety.



Rem 6.2 In general, X is singular in Def 6.1.

Zariski open set of X

Rem 6.3 $(U_X \subset X)$ is a toroidal embedding

normal var

$\underset{\text{def}}{\iff}$ it is locally isomorphic to

(analytic locally or formally)

$(T \subset V)$

dense torus

toric variety

- $f : X \longrightarrow Y$: toroidal

$\underset{\text{def}}{\iff}$ locally isomorphic to a morphism between toric varieties.

X, Y are both toroidal

We recall some basic results.

Lem 6.4 $f: X \to Y$: weakly semistable
$\Rightarrow$ X has only $\boxed{\text{rational}}$ / Gorenstein singularities.

(∵) X is locally isomorphic to a toric variety $\Rightarrow$ X: rational singularities

X: Gorenstein ← this part is nontrivial.

Lem 6.5 $f: X \to Y$: weakly semistable
$\Rightarrow$ f is flat.

(∵) X: Cohen-Macaulay, Y: smooth
f: equidimensional
$\Rightarrow$ f: flat
"

<u>Def 6.6</u> X: normal variety s.t

K_X: $\mathbb{Q}$-Cartier div $\Leftrightarrow$ [mK_X: Cartier for some $m \in \mathbb{Z}_{>0}$

$f: Y \to X$: resolution with

$$K_Y = f^* K_X + \sum_i a_i E_i$$

If $a_i \geq 0$ for ${}^\forall$resolution f and ${}^\forall i$

$\underset{\text{def}}{\Leftrightarrow}$ X has only "canonical singularities"

<u>Lem 6.7</u> X has only rational Gorenstein singularities

$\Leftrightarrow$ X has only canonical Gorenstein singularities.

Rem 6.8 X: normal variety.

$^\exists f: Y \longrightarrow X$ resolution such that

$$R^i f_* \mathcal{O}_Y = 0 \text{ for } {}^\forall i > 0.$$

Then we say that X has only rational singularities.

homework

Exercise 6.9 X: normal variety

X has only rational singularities

$$\Longleftrightarrow \begin{cases} \cdot\ X \text{ is Cohen-Macaulay} \\ \cdot\ {}^\exists f: Y \longrightarrow X \text{ resolution such that} \\ \qquad f_* \omega_Y = \omega_X. \end{cases}$$

Prove the above equivalence.

Exercise 6.10 (Lem 6.7)

X: normal variety with only Gorenstein singularities

Then

X has only rational singularities

$\Longleftrightarrow$ X has only canonical singularities

Prove the above equivalence.

Thm 6.11 (Abramovich-Karu)

↑ weak semistable reduction theorem

$f : X \to Y$: surj morphism between smooth projective varieties with connected fibers.

$\Rightarrow {}^{\exists} Y' \to Y$: generically finite surj morphism from a smooth proj variety Y'

${}^{\exists} X' \to X \times_Y Y'$: proj bir morphism from a normal proj variety X' s.t $X' \to Y'$ is weakly semistable

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

any map

gen. finite.

weakly s.s

"

The proof of Th 6.8 depends on de Jong's idea about alteration."

The following lemma is very important.

Lem 6.12 $f : X \to Y$: weakly semistable

$\tau : Y' \to Y$: morphism from a smooth projective variety Y' s.t $\tau^{-1}(Y \setminus U_Y)$ is a s.n.c div on Y'

$\Longrightarrow$

$$\begin{tikzcd} X' \arrow[r, "\rho"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r, "\tau"'] & Y \end{tikzcd}$$

weakly s.s.

$X' := X \times_Y Y'$

$f' : X' \to Y'$ is also weakly semistable."

Conjecture 6.13 $\quad X, Y$: projective

$f: X \longrightarrow Y$: surjective morphism with connected fibers

Assume that $f: X \longrightarrow Y$: weakly semistable

$\Longrightarrow f_* \omega_{X/Y}^{\otimes m}$: locally free for $\forall m \geqq 1$

X : normal var rational Gorenstein

Y : smooth

Thm 6.14 (Fujino)

$f: X \to Y$ as in Conj 6.13

[We further assume that F has a good minimal model, where]

F is a general fiber of $f: X \to Y$.

$\Rightarrow$ $f_* \omega_{X/Y}^{\otimes m}$: nef locally free sheaf for $\forall m \geq 1$.

Rem 6.15 ◎ local freeness of $f_* \omega_{X/Y}^{\otimes m}$ is nontrivial. nefness is easy to prove.

- $\mathrm{Var}(f) = \dim Y$
- F : of general type

$\Longrightarrow$ $\exists k > 0$ s.t. $f_* \omega_{X/Y}^{\otimes k}$: big sheaf