Lecture 3　2022.8.11

§7 $f_* \omega_{X/Y}^{\otimes m}$ for weakly s.s morphisms

Today's first goal is to prove:

X: normal proj var

Thm 7.1 $f : X \to Y$: surjective morphism with connected fibers

smooth proj var

Assume ① $f : X \to Y$: weakly s.s

② F has a good minimal model, where F is a general fiber of $f : X \to Y$

$\Rightarrow$ $\overbrace{f_* \omega_{X/Y}^{\otimes m}}^{\text{invertible sheaf}}$: nef locally free sheaf on Y for $\forall m \geq 0$ "

$f : X \to Y$: weakly s.s
$\Rightarrow$ X: Gorenstein

$m = 0 \Rightarrow f_* \mathcal{O}_X \simeq \mathcal{O}_Y$

$m = 1 \Rightarrow f_* \omega_{X/Y}$

Proof of Thm 7.1

- **Step 1** In this step, we will prove that $f_*\omega_{X/Y}^{\otimes m}$ is nef under the assumption that it is $\boxed{\text{locally free}}$.

We consider $f : X \to Y$: weakly s.s

$$f^s : \underbrace{X \times_Y X \times_Y \cdots \times_Y X}_{s\text{-times}} \longrightarrow Y$$

s-fold fiber product of $f : X \to Y$.

For simplicity, we consider the case where $s = 2$.

f : flat

$$\begin{tikzcd} X \times_Y X \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

weakly s.s

$\left[\begin{array}{l} \checkmark X : \text{Gorenstein} \\ \checkmark f : \text{flat} \end{array}\right) \leftarrow \underline{f : X \to Y}\checkmark$

weakly s.s.

By the flat base change theorem.

$$\underline{\omega_{X \times_Y X / X}} \simeq \underline{p^* \omega_{X/Y}}$$

invertible sheaves

$\therefore$ $\checkmark \underline{X \times_Y X}$: Gorenstein

In particular, $X \times_Y X$ is $\underline{\text{CM}}$

Cohen-Mac...

Since $f : X \to Y$: weakly s.s.

we can check that $\underline{\underline{X \times_Y X}}$ is

smooth in codimension one.

$\Rightarrow \checkmark$ ~~X~~ $X \times_Y X$ is normal.

Moreover, we can check that

$\checkmark X \times_Y X$: toroidal.

In general, the normalization of $X \times_Y X$ is toroidal.

In particular, $X \times_Y X$ has only rational Gorenstein singularities.

By similar argument,

$X^{\Delta} := X \times_Y X \times_Y \cdots \times_Y X$: normal

Gorenstein, rational sing.

Anyway, X^{Δ} is singular in general.

However, X^{Δ} has only mild singularities.

We can apply the argument explained before to X^{Δ}.

Claim

$f_*^{s} \omega_{X^s/Y}^{\otimes m} \simeq \bigotimes^{s} f_* \omega_{X/Y}^{\otimes m}$

holds for $\forall s \geqq 1$.

The proof of Claim in Thm 4.1 works in the above setting.

Note that we assumed that

$f_* \omega_{X/Y}^{\otimes m}$: locally free.

Let us recall Popa-Schnell's theorem.

Th 7.2 (Th 5.9) (Popa-Schnell)

$f : X \to Y$: surj mor

X : $\boxed{\text{smooth proj}}$ var $/\mathbb{C}$

Y : proj var, $\dim Y = n$

$\mathcal{L}$: ample line bundle on Y s.t

$|\mathcal{L}|$: free

$k \in \mathbb{Z}_{>0}$

$\Rightarrow$ $f_* \omega_X^{\otimes k} \otimes \mathcal{L}^{\otimes l}$: globally generated

for $\forall l \geq k(\underset{\dim Y}{\underset{\shortparallel}{n}}+1)$ "

Rem 7.3 Th 7.2 holds even when X has only rational Gorenstein singularities.

$(\because)$ X has rational $\underline{\text{Gorenstein}}$ singularities.

by Hironaka's resol

Let $g : Z \to X$: any resolution.

Then $\quad \underline{\underline{K_Z}} = g^* \underline{\underline{K_X}} + \underline{\sum \underset{\boxed{\overset{\vee\shortparallel}{0}}}{a_i} E_i}$

Cartier $\quad$ rational Gorenstein

E_i : g-exceptional. $\quad a_i \in \mathbb{Z}_{\geq 0}$

$\Rightarrow \checkmark\; g_* \omega_Z^{\otimes k} \simeq \omega_X^{\otimes k}$

smooth proj

By applying $\underline{\text{Th 7.2}}$ to $\underline{f \circ g : Z \to Y}$,

$(f \circ g)_* \omega_Z^{\otimes k} \otimes \mathcal{L}^{\otimes l}$ is globally generated for $l \geq k(n+1)$.

Note that $\quad (f \circ g)_* \omega_Z^{\otimes k} \underline{\underline{\simeq}} f_* \omega_X^{\otimes k}$

This is what we wanted. //

Let's go back to the proof of Th 7.1.

We put $\mathcal{M} = \omega_Y^{\otimes m} \otimes \mathcal{L}^{\otimes m(\dim Y+1)}$

$\mathcal{L}$: ample s.t $|\mathcal{L}|$: free.

By Th 7.2 (Rem 7.3),

X^s: rational Gorenstein

$$f^s : X^s \to Y$$

$f^s_* \omega_{X^s/Y}^{\otimes m} \otimes \mathcal{M}$: globally generated for $^\forall s \geq 1$

$\wr\|$ ← Claim

$\mathcal{M}$: independent of s

- $\overset{s}{\otimes}(f_* \omega_{X/Y}^{\otimes m}) \otimes \mathcal{M}$

By Lem 4.11, $f_* \omega_{X/Y}^{\otimes m}$: nef locally free.

We finish Step 1 in the proof of Th 7.1. //

Hence, all we have to do is to prove the local freeness of $f_*\omega_{X/Y}^{\otimes m}$ under the assumption that

① $f: X \longrightarrow Y$: weakly s.s.

② F has a good minimal model

$\uparrow$

general fiber of $f: X \to Y$.

Let us quickly explain good minimal models.

Def 7.4 $f : X \to Y$: projective morphism
X, Y: quasi-projective varieties.

✓ X has only canonical singularities
(terminal singularities)

X' in the following diagram

$$\begin{tikzcd} X \arrow[rr, dashed, "\phi\ (\text{bir})"] \arrow[dr, "f\ (\text{proj})"'] & & X' \arrow[dl, "f'\ (\text{proj})"] \\ & Y & \end{tikzcd}$$

is called a minimal model of X over Y if

X': normal quasi-proj variety

(i) X': $\mathbb{Q}$-factorial

(ii) f': projective

(iii) ϕ: birational, $\phi^{-1}: X' \dashrightarrow X$

ϕ^{-1} has no exceptional divisors

✓(iv) $K_{X'}$: f'-nef

✓(v) $a(E, X) < a(E, X')$ ✓

holds for $\forall \phi$-exceptional divisor

E on X

Moreover, if $K_{X'}$ is f'-semiample

$\Rightarrow$ X' is called a good minimal model of X over Y.

Rem 7.5 On (i),

X': $\mathbb{Q}$-factorial $\underset{\text{def}}{\iff}$ for ${}^\forall$div D (Weil div),
$\exists m \in \mathbb{Z}_{>0}$ s.t
mD: Cartier.

It is well known that if X' has only quotient singularities then X' is $\mathbb{Q}$-factorial.

On (iv),

$K_{X'}$ is f'-nef (numerically effective)

$\underset{\text{def}}{\iff}$ $K_{X'} \cdot C \geqq 0$ for ${}^\forall$projective curve C on X' such that $f'(C)$ is a point.

$K_{X'}$ (Weil div) is f'-semiample

$\underset{\text{def}}{\iff}$ $\exists m \in \mathbb{Z}_{>0}$ s.t $\underline{mK_{X'} \text{ is Cartier}}$ and $f'^* f'_* \underline{\mathcal{O}_{X'}(mK_{X'})} \twoheadrightarrow \mathcal{O}_{X'}(mK_{X'})$

[$K_{X'}$: f'-sa $\Rightarrow$ $K_{X'}$: f'-nef] well-defined Weil div up to $\sim$

Def 7.6 X : normal variety such that K_X is $\mathbb{Q}$-Cartier

(This means that $\exists m \in \mathbb{Z}_{>0}$ s.t mK_X is Cartier.)

Let $f : Y \to X$ be any resolution.

Then we can write

$$mK_Y = f^* mK_X + \boxed{\sum_E m \underset{\mathbb{Q}}{\underset{\cap}{a(E, X)}} E}$$

E : f-exceptional divisor

$a(E, X)$ is called "discrepancy".

terminal $\to$ canonical

✓ $a(E, X) \geq 0$ for ${}^\forall f : Y \to X$, ${}^\forall E$

$\underset{\boxed{\text{def}}}{\iff}$ X has only canonical singularities.

✓ $a(E, X) > 0$ for ${}^\forall f : Y \to X$, ${}^\forall E$

$\underset{\boxed{\text{def}}}{\iff}$ X has only terminal singularities.

Example 7.7 $\dim X = 2$, that is,

X is an normal algebraic surface.

X : terminal singularities

$\iff$ X : smooth surface

✓ X : canonical singularities

$\iff$ X : smooth, or has only rational double points.

ADE singularities

homework

Exercise 7.8 Please check:

① X : smooth surface

$\Longrightarrow X$: terminal singularities

② X has only ADE singularities

$\Longrightarrow X$: canonical singularities

dim $X = 2$

③ X : terminal singularities

$\Longrightarrow X$: smooth surface

④ X : canonical singularities

$\Longrightarrow X$ has at worst ADE singularities

F : general fiber of $f : X \to Y$

<u>Conj 7.9</u> $f : X \to Y$: projective morphism

X, Y : quasi-projective varieties.

X : smooth $\left(\Rightarrow X \text{ is } \mathbb{Q}\text{-factorial}, \; X \text{ has only terminal singularities} \right)$

Assume that K_F is pseudo-effective.

$\Uparrow$ obvious $\quad$ "||." $\quad \Downarrow$ $\boxed{\text{Nonvanishing Conjecture}}$

$\kappa(F) \geqq 0$

$$\Rightarrow \quad \exists \quad \begin{tikzcd} X \arrow[rr, dashed, "\phi"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & Y & \end{tikzcd}$$

X' is a minimal model of X over Y.

<u>Conj 7.10</u> (Abundance Conjecture)

$K_{X'}$ is f'-semiample "

$\Longleftrightarrow$ X' is a good minimal model over Y.

homework

Exercise 7.11 (2-dimensional abundance)

X : smooth projective surface

K_X : nef

X itself is a minimal model

$\Longrightarrow K_X$: semi-ample

$$\begin{tikzcd} X \arrow[rr, equal] \arrow[dr] & & X \arrow[dl] \\ & \operatorname{Spec} \mathbb{C} & \end{tikzcd}$$

K_X : nef

We have to check

✓ K_X : nef $\Longrightarrow \chi(X) \geq 0$.

K_X : nef and big $\Longrightarrow K_X$: semiample

KS basepoint-freeness

$\chi(X) = 0 \Longrightarrow$ We can use classification table... .

Conjectures 7.9 and 7.10 are still open.
$\exists$ partial reslts.

Thm 7.12 $f : X \to Y$: proj morphism between normal quasi-proj varieties with connected fibers.

Assume
- X has only canonical singularities
- F has a good minimal model, where F is a general fiber of $f : X \to Y$.

$$\begin{tikzcd} F \arrow[rr, dashed] \arrow[dr] & & F' \arrow[dl] \\ & \mathrm{Spec}\,\mathbb{C} & \end{tikzcd}$$

F': good minimal model

$\Longrightarrow$ $\exists$
$$\begin{tikzcd} X \arrow[rr, dashed, "\phi"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & Y & \end{tikzcd}$$

X' : good minimal model of X over Y

The proof of 7.12 is not so easy.
It depends on the recent developments
of MMP after BCHM.

(Birkar-Hacon-Cascini-McKernan)

Thm 7.13 (BCHM) 2006

✓ X : smooth projective variety

(or. projective variety

with only canonical singularities)

Assume that X is of general type.

($\underset{\text{def}}{\Longleftrightarrow}$ $\kappa(X) = \dim X$)

$\Longleftrightarrow$ K_X : big

$\Rightarrow$ X has a good minimal model.

flips or divisorial cont

$X =: X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_m$

X_m : good minimal model

K_{X_m} : semi-aple.

$\Uparrow$ ← KS basepnt-free

K_{X_m} : nef and big

<u>Cor 7.14</u>: $f : X \to Y$: proj. morphism

X, Y : quasi-projective varieties

X has only canonical singularities

F : general fiber of $f : X \to Y$.

Assume that F is of general type.

$\Leftrightarrow \kappa(F) = \dim F$

$\Rightarrow \exists$

$$\begin{tikzcd} X \arrow[rr, dashed, "\phi"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & Y & \end{tikzcd}$$

X' is a good minimal model of X over Y.

Our goal is:

f: proj toroidal
$^\forall$fiber is reduced equidimensional

Th 7.15 (Th 7.1)

$f: X \to Y$: weakly s.s morphism with connected fibers

✓ F: has a good minimal model

$\Rightarrow$ $f_* \omega_{X/Y}^{\otimes m}$ is locally free for $^\forall m \in \mathbb{Z}_{>0}$.

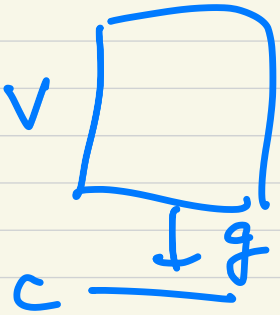
X: rational Gorenstein
$\Leftrightarrow$ X: canonical Gorenstein

Y: smooth proj var

F: general fiber of $f: X \to Y$

Main ingredient

Lem 7.16 $g : V \to C$: proj surjective morphism (C: curve)



V: normal quasi-proj variety

C: quasi-proj. smooth curve

Assumption: ✓ V has only canonical singularities

✓ K_V is g-semiample

$g : V \to C$ is a good minimal model of V itself over C

$\Longrightarrow$ $R^i g_* \boxed{\mathcal{O}_V(mK_V)}$: locally free for $\forall i$ and $\forall m \geq 1$.

$\overset{\text{equiv}}{\Longleftrightarrow}$ "torsion-free"

↑

This is due to Nakayama. (Noboru)

[Note that K_V is $\mathbb{Q}$-Cartier Weil divisor.

Lem 7.16 is essentially due to Kollár.

Thm 7.17 (Kollár's torsion-freeness)

$f : X \to Y$: projective surjective morphism

X : smooth　　X, Y : (quasi-)proj

$\Longrightarrow$ $R^i f_* \omega_X$: torsion-free for $\forall i$.

It is a generalization of the Grauert-Riemenschneider vanishing theorem.

Cor 7.18 $f : X \to Y$: projective birational morphism.

X : smooth

$\Longrightarrow$ $R^i f_* \omega_X = 0$ for $\forall i > 0$.

($\because$) $R^i f_* \omega_X$ is torsion for $\forall i > 0$.

By Thm 7.17, $R^i f_* \omega_X = 0$ for $\forall i > 0$.

By using the usual covering trick, we can prove

$$R^i f_*(\omega_X \otimes \underline{\underline{\mathcal{L}}})$$ ✓

is torsion-free for $\forall i > 0$ in Th 7.17, where $\mathcal{L}$ is an f-semiample line bundle.

In Lem 7.16,

$$\underline{mK_V} = K_V + \underbrace{\underline{(m-1)K_V}}_{\substack{\uparrow \\ g\text{-semiample}}}$$

by assumption

Note that $\underline{(m-1)K_V}$ is a $\mathbb{Q}$-Cartier Weil divisor. It is not necessarily Cartier.

However we can check that

$$R^i f_* \mathcal{O}_V(mK_V) : \text{torsion-free for } \forall i$$

in Lem 7.16.

Since C is a smooth curve,

$$R^i f_* \mathcal{O}_V(mK_V) : \text{locally free for } \forall i$$

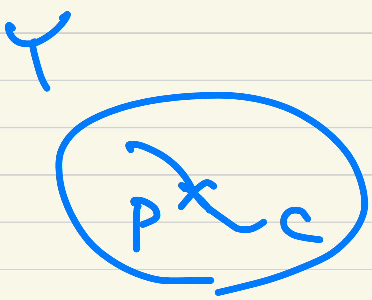
Tomorrow. I will first prove Th 7.15.

<u>Idea</u>

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & Y & \end{tikzcd}$$

$\leftarrow$ good minimal model of X/Y

$Y \ni {}^{\forall} p$ $\quad$ p: <u>any</u> point



$$\begin{tikzcd} X_C \arrow[r, hook] \arrow[d] & X' \arrow[d, "f'"] \\ C \arrow[r, hook] & Y \end{tikzcd}$$

We apply <u>Lem 7.16.</u>

[$f' : X' \to Y$ behaves well by base change.

We further assume

$$\left[\begin{array}{l} \kappa(F) = \dim F \iff F : \text{of gen type.} \\ \\ \mathrm{Var}(f) = \dim Y \end{array}\right.$$

$\uparrow$

Viehweg's variation

$\Longrightarrow\ ^{\exists}k > 0$ s.t

$f_* \omega_{X/Y}^{\otimes k}$: nef and $\underline{\underline{\text{big}}}$ locally free sheaf.

$$\left(\begin{array}{l} \omega_F : \text{ample} \\ \qquad \omega_F^{\underline{\underline{\otimes k}}} \end{array} \right)$$