Lecture 4 2022. 8. 12

§8 Proof of Thm 7.1

$f : X \to Y$: weakly semistable / $\mathbb{C}$

F : general fiber of f

☆ Assume F has a good minimal model

$\Rightarrow$ $\boxed{f_*\omega_{X/Y}^{\otimes m} \text{ is locally free}}$ for $\forall m \geqq 0$

$\Rightarrow$ $f_*\omega_{X/Y}^{\otimes m}$: nef for $\forall m \geqq 0$

we have already checked this part.

Proof

$$\begin{tikzcd} X \arrow[rr, dashed, "\phi"] \arrow[dr, "f"'] & & \tilde{X} \arrow[dl, "\tilde{f}"] \\ & Y & \end{tikzcd}$$

weakly s.s

$\tilde{X}$: good minimal model of X over Y

By Thm 7.12 + ☆, we have the above diagram.

$f : X \to Y$: weakly s.s

Note that X has only rational Gorenstein singularities

$$\Updownarrow$$

X has only canonical Gorenstein singularities

$$\Downarrow$$

$\tilde{X}$ has only canonical singularities

$$\Longrightarrow \tilde{f}_* \mathcal{O}_{\tilde{X}}(m K_{\tilde{X}/Y}) \simeq f_* \omega_{X/Y}^{\otimes m},$$

where $K_{\tilde{X}/Y} = K_{\tilde{X}} - \tilde{f}^* K_Y$

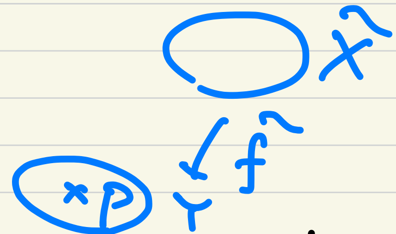
$\mathbb{Q}$-Cartier Weil div.

Cartier

K_Y : canonical divisor of Y

Hence it is sufficient to prove that ✓ $\tilde{f}_* \mathcal{O}_{\tilde{X}}(mK_{\tilde{X}/Y})$ is locally free.

We take any point $P \in Y$.



$P \in C = H_1 \cap \cdots \cap H_{n-1}$ $\quad n = \dim Y$

$P \in H_i$: general very ample Cartier divisors for $\forall i$

$P \in C$: a smooth curve

We consider

weakly s.s morphism

$$\begin{tikzcd} X_c \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ C \arrow[r, hook] & Y \end{tikzcd}$$

$X_c \to C$ is the base change of $f: X \to Y$ by $C \hookrightarrow Y$

By Lem 6.12, $X_c \to C$ is also weakly semistable.

$X_c \to C$ weakly s.s

In particular, X_c : normal,

X_c has only rational Gorenstein singularities.

We consider

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & X_{H_1} \arrow[l, hook'] \arrow[d] \arrow[dr, phantom, "\square"] & X_{H_1 \cap H_2} \arrow[l, hook'] \arrow[d] & \cdots \arrow[l, hook'] \arrow[dr, phantom, "\square"] & X_c \arrow[l, hook'] \arrow[d] \\ Y & H_1 \arrow[l, hook'] & H_1 \cap H_2 \arrow[l, hook'] & \cdots \arrow[l, hook'] & C \arrow[l, hook'] \end{tikzcd}$$

$P \in Y$

$C = H_1 \cap \cdots \cap H_{n-1}$

By Lem 6.12 and the fact that f is flat,

X_{H_1}, $X_{H_1 \cap H_2}$, $\cdots$, $X_{H_1 \cap \cdots \cap H_{n-2}}$

are all normal.

[they are normal outside $f^{-1}(P)$

codim $f^{-1}(P) \geq 2$

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The following lemma is well known.

Lem 8.1 $(\boxed{X_{H_1 \cap \cdots \cap H_{n-2}}}, X_c)$ is purely log terminal in a neighborhood of X_c

normal variety

Cartier div

Proof X_c has only rational Gorenstein singularities $\Rightarrow$ X_c : canonical singularities

$\Rightarrow$ X_c is (Kawamata) log terminal

Since X_c is Cartier in $X_{H_1 \cap \cdots \cap H_{n-2}}$, $(X_{H_1 \cap \cdots \cap H_{n-2}}, X_c)$ is purely log terminal (in a nbd of X_c) by Inversion of adjunction.

Since $X_{H_1 \cap \cdots \cap H_{n-2}}$ is Gorenstein,

$\Rightarrow$ $\boxed{X_{H_1 \cap \cdots \cap H_{n-2}}}$ has only canonical Gorenstein singularities.

By repeating this argument,

$$\left(X_{H_1 \cap \cdots \cap H_{n-3}}, \boxed{X_{H_1 \cap \cdots \cap H_{n-2}}} \right)$$

is purely log terminal in a neighborhood of $X_{H_1 \cap \cdots \cap H_{n-2}}$ by Inversion of adjunction.

$\Rightarrow X_{H_1 \cap \cdots \cap H_{n-3}}$: has only canonical Gorenstein singularities.

Repeat this finitely many times!

$$X_{H_1}, \ X_{H_1 \cap H_2}, \ \cdots\cdots, \ X_c$$

have only canonical Gorenstein singularities and

$\boxed{(X, X_{H_1})}$ has only purely log terminal singularities.

Next, we consider :

$$\begin{tikzcd}
\tilde{X} \arrow[d, "\tilde{f}"'] \arrow[r, bend left, phantom] & \tilde{X}_{H_1} \arrow[l, hook'] \arrow[d] \arrow[dl, phantom, "\square"] & \tilde{X}_{H_1 \cap H_2} \arrow[l, hook'] \arrow[d] \arrow[dl, phantom, "\square"] & \cdots & \tilde{X}_C \arrow[l, hook'] \arrow[d] \arrow[dl, phantom, "\square"] \\
Y & H_1 \arrow[l, hook'] & H_1 \cap H_2 \arrow[l, hook'] & \cdots & C \arrow[l, hook']
\end{tikzcd}$$

By negativity lemma,

✓ $\boxed{(\tilde{X}, \tilde{X}_{H_1})}$ is purely log terminal.

Moreover, we can check that

$\boxed{\tilde{X}_{H_1}}$ has only canonical singularities.

Apply the same argument to

✓ $(\boxed{X_{H_1}}, \boxed{X_{H_1 \cap H_2}})$, which is purely log terminal.

Then $(\widetilde{X}_{H_1}, \widetilde{X}_{H_1 \cap H_2})$ is also purely log terminal and

$\underline{\widetilde{X}_{H_1 \cap H_2}}$ has only canonical singularities.

Finally, we obtain

$\widetilde{X}_c$ is normal and has only canonical singularities.

We consider

normal variety

smooth proj curve

$$\widetilde{X}_c \longrightarrow C \ : \text{ dominant map}$$

$\Rightarrow$ It is flat.

$\Rightarrow$ $\underline{\widetilde{X}_c \to C}$: equidimensional fibers.
$\quad\quad\quad\quad\ \ \Downarrow$
$\quad\quad\quad\quad\ \ P$

$\Rightarrow$ $\underline{\widetilde{f} : \widetilde{X} \to Y}$: equidimensional fibers.

Note that $\widetilde{X}$: canonical singularities

$\Rightarrow \widetilde{X}$: rational singularities

$\Rightarrow \widetilde{X}$: Cohen-Macaulay

Hence $\widetilde{f} : \underset{\text{CM}}{\underline{\underline{\widetilde{X}}}} \to \underset{\text{smooth.}}{\underline{\underline{Y}}}$ is flat.

$P \in Y$

Since $\tilde{X}$ has only canonical singularities, $\mathcal{O}_{\tilde{X}}(mK_{\tilde{X}})$: Cohen-Macaulay for $\forall m \in \mathbb{Z}$.

In particular, $\mathcal{O}_{\tilde{X}}(mK_{\tilde{X}})$: flat $/Y$.

$K_{\tilde{X}_c/C}$: semi ample/C

By adjunction,

$$\mathcal{O}_{\tilde{X}}(mK_{\tilde{X}/Y})|_{\tilde{X}_c} \simeq \boxed{\mathcal{O}_{\tilde{X}_c}(mK_{\tilde{X}_c/C})}$$

By Lem 7.16 and the flat base change theorem, ↑Nakayama

✓ $\dim_{\mathbb{C}} H^0(\tilde{X}_y, \mathcal{O}_{\tilde{X}}(mK_{\tilde{X}/Y})|_{\tilde{X}_y})$

is independent of $y \in Y$.

normal var

$R^i f_* -$ is locally for $\forall i$

✓ $\tilde{X}_c \to C$ smooth proj curve

✓ $K_{\tilde{X}_c/C}$: rel. s. ample

This implies

$$\underline{\underline{\tilde{f}_* \mathcal{O}_{\tilde{X}}(mK_{\tilde{X}/Y})}} : \text{locally free sheaf for } {}^\forall m \geq 1$$

$$\wr\|$$

Thus $f_* \mathcal{O}_X(mK_{X/Y})$ is locally free sheaf for ${}^\forall m \geq 1$

$$\| \quad \underline{f_* \omega_{X/Y}^{\otimes m}}$$

Hence we finish the proof of Th 7.1.

homework

Exercise 8.2 (Singularities of pairs)

X: normal variety $/\mathbb{C}$

Δ: effective $\mathbb{Q}$-divisor on X

s.t $K_X + \Delta$: $\mathbb{Q}$-Cartier

$\Delta \in [0, 1]$

① Define Kawamata log terminal, purely log terminal, and log canonical singularities for (X, Δ).

(Adjunction, Inversion of adjunction)

② S: irreducible component of $\lfloor \Delta \rfloor$ s.t S is Cartier

We put $K_S + \Delta_S := (K_X + \Delta)|_S$

$\Rightarrow$ (S, Δ_S): kawamata log terminal

$\Leftrightarrow$ (X, Δ): purely log terminal in a neighborhood of S.

$\Leftarrow$ adjunction

$\Rightarrow$ Inversion of adjunction

§9 Towards Thm 3.1

Thm 9.1 $\left[\begin{array}{l} f: X \to Y: \text{ weakly semistable morphism} \\ f \text{ has connected fibers} \end{array}\right.$

F: of general type $\quad \kappa(F) = \dim F$

↑ general fiber of $f: X \to Y$

Assumption ✓ $\text{Var}(f) = \dim Y$

$\Rightarrow \exists k > 0$ such that $\boxed{f_* \omega_{X/Y}^{\otimes k}}$ is a nef and big locally free sheaf on Y

$\left[\begin{array}{l} F: \text{ of general type} \\ \quad \Rightarrow F \text{ always has a good minimal model} \\ \quad \uparrow \\ \text{BCHM} \end{array}\right.$

Let us recall Viehweg's variation.

Def 9.2 (Viehweg's variation)

$f: X \to Y$: surjective morphism of normal (quasi-)proj. varieties $/\mathbb{C}$

$K (\supset \mathbb{C})$: algebraically closed field contained in $\overline{\mathbb{C}(Y)}$

$\mathbb{C} \subset K \subset \overline{\mathbb{C}(Y)}$

s.t $\exists$ smooth proj var V defined over K and

$\boxed{V \times_{\mathrm{Spec}\,K} \mathrm{Spec}\,\overline{\mathbb{C}(Y)}}$ and $\boxed{X \times_Y \mathrm{Spec}\,\overline{\mathbb{C}(Y)}}$

$\mathbb{C} \subset \overline{\mathbb{C}(Y)}$

are birational. ✓

$\mathrm{Var}(f) :\underset{\mathrm{def}}{=} \min_{K} \mathrm{trans.deg}_{\mathbb{C}} K$

↑

It is called the variation of f.

$0 \leq \mathrm{Var}(f) \leq \dim Y$.

If $\mathrm{Var}(f)=0$

$\Rightarrow$ roughly speaking,

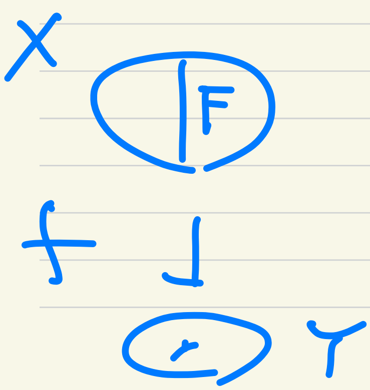
$f: X \to Y$ is trivial, that is,

$$\begin{array}{c} X = Y \times F \\ \downarrow \\ Y \end{array}$$

f is birationally equivalent to isotrivial fibration.

If $\underline{\mathrm{Var}(f) = \dim Y}$

$\Rightarrow$ F deforms nontrivially in any direction.



Conj 9.3 (Viehweg's Conj Q)

$f : X \to Y$: surj mor between smooth proj varieties with connected fibers

Assume $\kappa(F) \geq 0$

$\uparrow$ general fiber of f

Assumption : $\underline{\mathrm{Var}(f) = \dim Y}$

$\Rightarrow$ $f_* \omega_{X/Y}^{\otimes k}$ is big for some positive integer k

✓ Conj Q $\Longrightarrow$ Iitaka conjecture

$\kappa(X) \geq \kappa(Y) + \kappa(F)$

Viehweg's (complicated) clever argument

Def 9.4 (Big sheaves)

$\mathcal{F}$: torsion-free coherent sheaf on a smooth projective variety W

$\mathcal{F}$: big

$\underset{\text{def}}{\iff}$ ${}^{\exists}\mathcal{H}$: ample line bundle on W ${}^{\exists}\nu \in \mathbb{Z}_{>0}$ s.t

$$\underbrace{\bigoplus_{\text{finite}} \mathcal{H}} \hookrightarrow \underbrace{\hat{S}^{\nu}(\mathcal{F})}$$

generically isomorphic injection

We note $\hat{S}^{\nu}(\mathcal{F}) \underset{\text{def}}{:=} (S^{\nu}(\mathcal{F}))^{**}$

We prove Thm 9.1 modulo the following lemma.

Lem 9.5 $f: X \to Y$: weakly semistable

f has connected fibers

Assume that $\boxed{f_* \omega_{X/Y}^{\otimes m}}$ is a nef locally free sheaf on Y.

Assume $\kappa(Y, \boxed{\det(f_* \omega_{X/Y}^{\otimes m})}) = \dim Y$

$\Longrightarrow$ ${}^{\exists} k > 0$ s.t

$\boxed{f_* \omega_{X/Y}^{\otimes k} \text{ is big}}$

Rem 9.6 In Lem 9.5,

$\det(f_* \omega_{X/Y}^{\otimes m})$ is a nef line bundle.

Hence

$$\chi(Y, \det(f_* \omega_{X/Y}^{\otimes m})) = \dim Y$$

$$\Updownarrow$$

$$\boxed{(\det(f_* \omega_{X/Y}^{\otimes m}))^{\dim Y} > 0}$$

$\uparrow$

the self-intersection number of $\det(f_* \omega_{X/Y}^{\otimes m})$

Rem 9.7

Viehweg's argument

① Thm 9.1 $\Longrightarrow$ Iitaka conjecture when F: general type.

Kollár's theorem (1987)

$f : X \longrightarrow Y$: surj mor

X. Y : smooth proj var

F : of general type

$\Rightarrow \kappa(X) \geqq \kappa(Y) + \kappa(F)$ "

② The following proof of Thm 9.1 is essentially Kollár's projectivity criterion of moduli spaces.

JDG

Proof of Thm 9.1

We consider

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[d, "f"'] & \widetilde{X} \arrow[dl, "\tilde{f}"] \\ Y & \end{tikzcd}$$

weakly s.s given

$\tilde{f} : \widetilde{X} \to Y$: relative canonical model.

More explicitly,

$$\widetilde{X} := \mathrm{Proj}_Y \bigoplus_{m \geq 0} f_* \mathcal{O}_X(mK_X)$$

- $\widetilde{X}$ has only canonical singularities
- $K_{\widetilde{X}}$: $\tilde{f}$-ample
- $f_* \omega_{X/Y}^{\otimes m} \simeq \tilde{f}_* \mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}/Y})$ for $\forall m \geq 0$.

  ref. locally free

Hence, it is sufficient to prove that

$$\left(\det \tilde{f}_* \mathcal{O}_{\widetilde{X}}(kK_{\widetilde{X}/Y})\right)^{\dim T} > 0$$

for some $k > 0$.

From now on, we write $f: X \to Y$ to denote $\tilde{f}: \tilde{X} \to Y$ for simplicity.

We fix $l \gg 0$ such that

$lK_{X/Y}$: f-very ample.

We further assume that

✓ (1) $\delta : \overbrace{S^{\mu}(f_* \mathcal{O}_X(lK_{X/Y}))}^{\mathcal{E}} \to f_* \mathcal{O}_X(\mu l K_{X/Y})$: surjective for $\forall \mu \in \mathbb{Z}_{>0}$.

($\because$) This is because $K_{X/Y}$: f-ample.

We put $\mathcal{E} := f_* \mathcal{O}_X(\ell K_{X/Y})$ and consider

nef locally free sheaf

$\ell K_{X/Y}$: f-very ample

$$\begin{tikzcd} X \arrow[rr, hook, "\iota"] \arrow[dr, "f"'] & & \mathbb{P}(\mathcal{E}) \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

$\mathcal{I}$: the defining ideal sheaf of $\iota(X)$ on $\mathbb{P}(\mathcal{E})$.

Then $\exists \mu \in \mathbb{Z}_{>0}$ such that

✓ (2) $p^* p_* (\mathcal{I} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(\mu)) \twoheadrightarrow \mathcal{I} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(\mu)$: surjective "

From now on, we fix this μ.

Rem 9.7 δ in (1) is nothing but the restriction map

$$p_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(\mu) \twoheadrightarrow f_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(\mu)|_X).$$

Hence $\operatorname{Ker}\delta = p_*(\mathcal{L}\otimes\mathcal{O}_{\mathbb{P}(\mathcal{E})}(\mu))$.

By (2), $\operatorname{Ker}\delta$ recovers $\mathcal{L}$.

Therefore, $\operatorname{Ker}\delta$ recovers

$f: X \to Y$. //

$$\mathcal{O}_{\mathbb{P}(\mathcal{E})}(\mu)|_X = \mathcal{O}_X(\mu\,\ell K_{X/Y})$$

$X = \imath(X)$

We put $\pi : \mathbb{P} := \mathbb{P}(\overset{r}{\oplus}\underline{\mathcal{E}}^*) \longrightarrow T$

$$r = \operatorname{rank} \mathcal{E}$$

Then we obtain the universal basis map

$$\underline{\underline{s}} : \overset{r}{\oplus} \mathcal{O}_{\mathbb{P}}(-1) \longrightarrow \pi^*\mathcal{E}$$

Δ : zero divisor of $\det(s)$.

We put $Q := f_* \mathcal{O}_X(\mu \ell K_{X/T})$

and consider

$$\delta : S^{\mu}(\mathcal{E}) \twoheadrightarrow \underset{\underset{f_* \mathcal{O}_X(\mu \ell K_{X/T})}{\parallel}}{Q}$$

$\underline{\mathcal{B}} \subset \pi^* Q$ is the image of the following map

$$\mathcal{S}^\mu\big(\underline{\bigoplus^r \mathcal{O}_{\mathbb{P}}(-1)}\big) \underline{=} \mathcal{S}^\mu\big(\bigoplus^r \mathcal{O}_{\mathbb{P}}\big) \otimes \mathcal{O}_{\mathbb{P}}(-\mu)$$

$$\xrightarrow[\mathcal{S}^\mu(s)]{} \mathcal{S}^\mu(\pi^* \mathcal{E}) \xrightarrow[\pi^* \underline{\underline{f}}]{} \pi^* Q$$

By taking some blow-ups which are isomorphisms outside Δ, we have

$$\tau : \mathbb{P}' \longrightarrow \mathbb{P} \ : \text{proj bir mor}$$

such that

$$\mathcal{B}' := \tau^* \mathcal{B} / \text{torsion}$$

is locally free.

$\pi' : \mathbb{P}' \xrightarrow{\tau} \mathbb{P} \xrightarrow{\pi} Y$

We put $\mathcal{O}_{\mathbb{P}'}(1) := \tau^*\mathcal{O}_{\mathbb{P}}(1)$

$$\pi' := \pi \circ \tau .$$

Then $\theta : \underline{S^{\mu}\left(\bigoplus^{r} \mathcal{O}_{\mathbb{P}'}(-1)\right)} \twoheadrightarrow \underline{\underline{\mathcal{B}'}}$: surjective

This $\underline{\theta}$ corresponds to : $\operatorname{rank}\mathcal{B}' = \operatorname{rank} Q$

$\rho' : \underline{\mathbb{P}'} \longrightarrow \mathrm{Grass}(\operatorname{rank} Q, S^{\mu}(\mathbb{C}^r))$

$$\hookrightarrow \mathbb{P}^M$$

$\uparrow$

Plücker embedding.

Then $\underline{\det(\mathcal{B}') \otimes \mathcal{O}_{\mathbb{P}'}(\gamma)} \simeq \rho'^* \mathcal{O}_{\mathbb{P}^M}(1)$

$$\gamma = \underline{\underline{\mu}} \cdot \underline{\operatorname{rank} Q} .$$

Lem 9.8 F: general fiber of $f : X \to Y$.

Then K_F : ample and

F has only canonical sing.

Thus $\mathrm{Aut}(F)$ is finite.

$\uparrow$

automorphism group of.

By this lemma, we can check that

$$\rho' : \mathbb{P}'_y := (\pi \circ \tau)^{-1}(y) \longrightarrow \rho'(\mathbb{P}'_y)$$

is generically finite, where

y is a general point of Y.

Lem 9.9 $\rho' : \mathbb{P}' \to \rho'(\mathbb{P}')$ is generically finite.

$Y' \subset \mathbb{P}'$

$$\begin{tikzcd} Y' \arrow[r, phantom, "\subset"] \arrow[dr] & \mathbb{P}' \arrow[d] \\ & Y \end{tikzcd}$$

(∵) We assume that $Y' \supset$

$$\rho' : \mathbb{P}' \to \rho'(\mathbb{P}')$$

is not generically finite.

Then we can take a general subvariety $Y' \subset \mathbb{P}'$ such that

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

$Y' \to Y$ is gen finite.

$\rho' : Y' \to \rho'(Y')$: not gen finite.

Let X' be a resolution of the main component of $X \times_Y Y'$.

Then $\dim Y = \mathrm{Var}(f) = \mathrm{Var}(f') < \dim Y'$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \dim Y.$

assuption $\qquad$ $Y' \to Y$: gen finite

This is a contradiction.

Since $\rho' : \mathbb{P}' \longrightarrow \rho'(\mathbb{P}') \subset \mathbb{P}^M$

is gen. finite, by Lem 9.9

$$\rho'^* \mathcal{O}_{\mathbb{P}^M}(1)$$

is nef and big on $\mathbb{P}'$.

By Kodaira's lemma,

$$H^0(\mathbb{P}', \boxed{\rho'^* \mathcal{O}_{\mathbb{P}^M}(\nu)} \otimes \pi'^* \mathcal{O}_Y(-H)) \neq 0$$

for some $\nu \gg 0$, where H is an ample divisor on Y.

Since π^*Q coincides with $\mathcal{B}'$ over a nonempty Zariski open set,

$$\pi'^*(\mathcal{O}_Y(-H)\otimes\det(Q)^\nu))\otimes\underbrace{\mathcal{O}_{\mathbb{P}'}(\nu\cdot\gamma)}_{\alpha}$$

has a section.

We put $\alpha=\nu\cdot\gamma$.

Then we get a nontrivial map

$$\varphi:(\pi'_*\mathcal{O}_{\mathbb{P}'}(\alpha))^*=S^\alpha\left(\bigoplus^r\mathcal{E}\right)$$

$$\longrightarrow\mathcal{O}_Y(-H)\otimes\det(Q)^\nu$$

By taking a birational modification $g:Y'\longrightarrow Y$, we have

$$\mathcal{G}\otimes\mathcal{O}_{Y'}(F)=g^*\mathcal{O}_Y(-H)\otimes g^*(\det Q)^\nu$$

where $F\geqq 0$ and $\mathcal{G}$ is a quotient line bundle of $g^*\left(S^\alpha\left(\bigoplus^r\mathcal{E}\right)\right)$.

Lem 9.10 $\mathcal{E}$: nef

$$\Rightarrow \underline{g}^*(\underline{S^\alpha}(\boxed{\bigoplus^r \mathcal{E}})) : \text{nef}$$

$\Rightarrow \underline{\mathcal{G}}$ is a nef line bundle //

Exercise 9.11 Prove Lem 9.10 //

We put $n = \dim Y = \dim Y'$.

$$\underset{\substack{\vee \\ 0}}{\boxed{((\det Q)^{\nu})^{n}}} = (\underbrace{g^{*}(\det Q)^{\nu}}_{\text{nef}})^{n}$$

$$= (\underbrace{g^{*}\mathcal{O}_{Y'}(H)}_{\text{nef}} \otimes g \otimes \underbrace{\mathcal{O}_{Y'}(F)}_{\text{eff}}) \cdot (\underbrace{g^{*}(\det Q)^{\nu}}_{\text{nef}})^{n-1}$$

$$\geqq \cdots$$

$$\geqq \cdots$$

$$\geqq H^{n} > 0 \quad \leftarrow \quad H : \text{ample on } Y$$

Hence $\det Q = \det f_{*}\mathcal{O}_{X}(\mu l K_{X/Y})$

is nef and big.

This is what we wanted.

We put $k = \mu l$

$\Rightarrow f_{*}\omega_{X/Y}^{\otimes k}$ is big.