

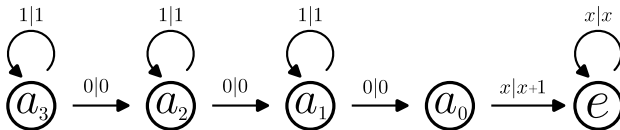
Mini course on automata and groups (part 3)

Eduard Schesler

Karlsruhe Institute of Technology

18. September 2026

Talk given at the Shanghai Center for Mathematical Sciences (SCMS).



Theorem (Grigorchuk's formulation (2000))

For a finitely generated just-infinite group G , one of the following holds:

- *G is a branch group (equivalently, $\mathcal{L}(G)$ is infinite).*
- *G is virtually a direct product of infinite simple groups.*
- *G has a normal finite-index subgroup isomorphic to L^k , where L is residually finite and every finite-index subgroup of L is just-infinite; that is, L is hereditarily just-infinite.*

Examples

- *Simple groups:* Thompson's groups T, V ; Burger–Mozes groups; certain Kac–Moody groups; certain lattices on exotic $\tilde{C}_2$ -buildings; ...
- *Branch groups:* Grigorchuk's group, Gupta–Sidki groups, ...
- *Hereditarily just-infinite groups:* $\mathbb{Z}$, D_∞ , $\mathrm{PSL}_n(\mathbb{Z})$ for $n \geq 3$; locally 0–1 groups (Ershov–Jaikin–Zapirain); representation-theoretic small-cancellation groups (Kionke–Schesler).

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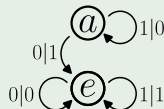
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Example (Binary adding machine)

The element a acts on X^ℓ as a cycle of length 2^ℓ .

Hence $a^{2^\ell} \in \text{St}_{\langle a \rangle}(\ell)$; in fact,

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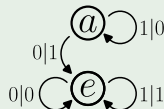
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Remark

The groups $\text{St}_G(\ell)$ form a residual chain. For $\langle a \rangle \cong \mathbb{Z}$, it is $\mathbb{Z} \geq 2\mathbb{Z} \geq 4\mathbb{Z} \geq \dots$

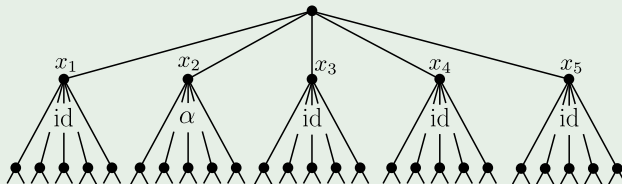
Definition (Rigid vertex stabilizer)

For each $G \leq \text{Aut}(T_X)$ and each $v \in X^*$, the *rigid vertex stabilizer* of v in G , denoted by $\text{RiSt}_G(v)$, is defined as the subgroup of $\text{St}_G(v)$ that fixes every word that does not start with v .

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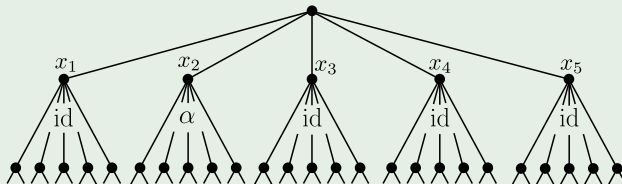


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Example

Consider the binary adding machine $a = \sigma \cdot (\text{id}, a)$. Then $\text{St}_{\langle a \rangle}(1) = \langle a^2 \rangle$, while $\text{RiSt}_{\langle a \rangle}(v) = 1$ for every $v \in X^*$:

$$a^2 = \sigma \cdot (\text{id}, a) \cdot \sigma \cdot (\text{id}, a) = (a, \text{id}) \cdot (\text{id}, a) = (a, a).$$

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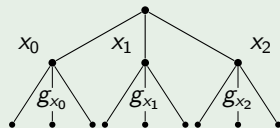
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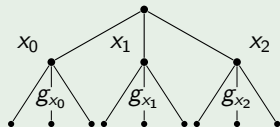
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$$\text{RiSt}_{\langle a \rangle}(v) = 1 \quad \forall v \in X^*, |v| > 0 \quad \implies \quad \text{RiSt}_{\langle a \rangle}(\ell) = 1 \quad (\ell \geq 1).$$

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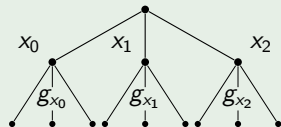
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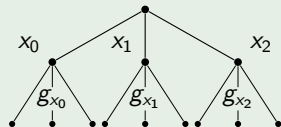
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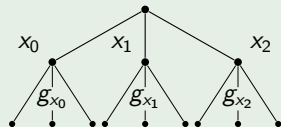
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Indeed, let $a^n \in \text{RiSt}_{\langle a \rangle}(v)$ with $|v| > 0$.

- a^n fixes a vertex of X^m for every $m > |v|$.
- a is a 2^m -cycle on X^m , so $2^m \mid n$ for all $m > |v|$; hence $n = 0$.

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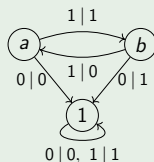
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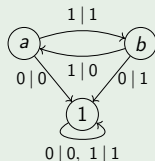
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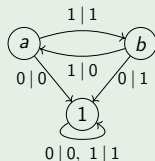
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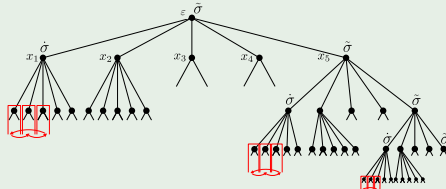
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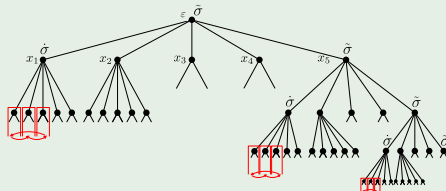


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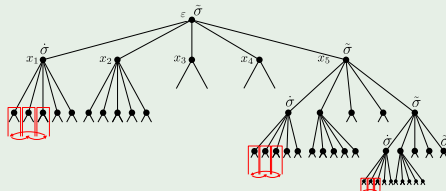
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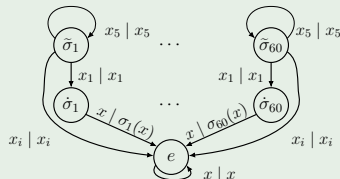
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Remark

Equivalently, Segal's group Γ_d is generated by

$$\dot{\sigma} = \sigma(1, \dots, 1) \text{ and } \tilde{\sigma} = (\dot{\sigma}, 1, \dots, 1, \tilde{\sigma}),$$

where $\sigma \in \text{Alt}(X)$.



Lemma (Segal)

The group $\Gamma_d = \langle \dot{\sigma}, \tilde{\sigma} = (\dot{\sigma}, 1, \dots, 1, \tilde{\sigma}) \mid \sigma \in \text{Alt}(X) \rangle$ is a branch group.

Proof: the commutator trick.

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$$\tilde{\sigma}_1 = (\underset{x_1}{\dot{\sigma}_1}, 1, \dots, 1, \underset{x_d}{\tilde{\sigma}_1}),$$

$$\tilde{\sigma}_2^{\dot{\tau}} = \left(\underset{x_1}{\dot{\sigma}_2}, 1, \dots, \underset{\tau^{-1}(x_d)}{\tilde{\sigma}_2}, 1, \dots, \underset{x_d}{1} \right).$$

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- Perfectness of $\text{Alt}(X)$ implies that these elements generate the claimed copy.

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The group $\Gamma_d = \langle \dot{\sigma}, \tilde{\sigma} = (\dot{\sigma}, 1, \dots, 1, \tilde{\sigma}) \mid \sigma \in \text{Alt}(X) \rangle$ is a branch group.

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$$\gamma' \in \text{St}_{\Gamma_d}(1): \quad \gamma'_x = \gamma, \quad \gamma'_y = 1 \quad (y \neq x),$$

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$$\text{Then } \tilde{\gamma}|_{X^{\ell+1}} = \text{id}, \quad \tilde{\gamma}_{wy} = (\tilde{\gamma}_w)_y = \begin{cases} 1, & w \neq v, \\ 1, & w = v, y \neq x, \\ \gamma, & w = v, y = x. \end{cases}$$

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 - Using $(gh)_v = g_{h(v)}h_v$, induction on word length gives

$$\gamma_v \in \Gamma_d \quad (\gamma \in \Gamma_d, v \in X^*).$$

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$$\gamma = (\gamma_v)_{v \in X^\ell} \in \Gamma_d^{X^\ell}.$$

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- Consequently,

$$\Gamma_d^{X^\ell} \leq \text{RiSt}_{\Gamma_d}(\ell) \leq \text{St}_{\Gamma_d}(\ell) \leq \Gamma_d^{X^\ell}.$$

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 - Set $\gamma = \gamma^{(1)}\gamma^{(2)}\dots\gamma^{(\ell)}$. Then $\gamma(x_1^\ell) = w$:

$$x_1^\ell \xrightarrow{\gamma^{(\ell)}} x_1^{\ell-1} x_{i_\ell} \xrightarrow{\gamma^{(\ell-1)}} \dots \xrightarrow{\gamma^{(1)}} x_{i_1} \dots x_{i_\ell} = w.$$

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- Γ_d acts transitively on every level X^ℓ .
- Since X^ℓ is finite and $\text{St}_{\Gamma_d}(\ell) = \ker(\Gamma_d \longrightarrow \text{Sym}(X^\ell))$, we obtain

$$[\Gamma_d : \text{RiSt}_{\Gamma_d}(\ell)] = [\Gamma_d : \text{St}_{\Gamma_d}(\ell)] < \infty.$$

Hence Γ_d is branch.

Definition (Word growth)

Let G be a finitely generated group with finite generating set S . The *word growth function* of G is given by

$$\gamma_G^S: \mathbb{N} \rightarrow \mathbb{N}, \quad n \mapsto \#\{g \in G \mid |g|_S \leq n\},$$

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Example

Let F_2 be a free group of rank 2 with basis S . Then

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Example

For the standard generating set S of $\mathbb{Z}^2$ we have

$$\gamma_{\mathbb{Z}^2}^S(n) = 2n^2 + 2n + 1.$$

Theorem (Gromov; 1981)

A finitely generated group has polynomial growth if and only if it is virtually nilpotent.

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Theorem (Grigorchuk; 1984)

There is a group $\mathcal{G}$ whose growth function satisfies

$$e^{\sqrt{n}} \leq \gamma_{\mathcal{G}}^S(n) \leq e^{n^\alpha} \quad \text{for some } \frac{1}{2} < \alpha < 1.$$

Theorem (Erschler, Zheng; 2020)

The growth function of Grigorchuk's group $\mathcal{G}$ satisfies

$$\lim_{n \rightarrow \infty} \frac{\log(\log(\gamma_{\mathcal{G}}^S(n)))}{\log(n)} = \alpha_0, \quad \alpha_0 = \frac{\log(2)}{\log(2) - \log(\lambda)},$$

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Conjecture (Grigorchuk's Gap Conjecture; 1990)

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Theorem (Shalom, Tao; 2010)

If G is not virtually nilpotent then $\gamma_G^X(n) \geq n^{c(\log \log n)^c}$.

Theorem (Erschler, Zheng; 2020)

The growth function of Grigorchuk's group $\mathcal{G}$ satisfies

$$\lim_{n \rightarrow \infty} \frac{\log(\log(\gamma_{\mathcal{G}}^S(n)))}{\log(n)} = \alpha_0, \quad \alpha_0 = \frac{\log(2)}{\log(2) - \log(\lambda)},$$

where λ is the positive real root of $X^3 + X^2 + X - 2$.

Conjecture (Grigorchuk's Gap Conjecture; 1990)

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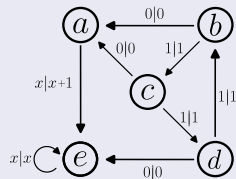
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Theorem (Eberhard, Maini, Sabatini, Tracey; 2026)

If every non-abelian finite simple group G satisfies $\text{diam}(G) \leq (\log |G|)^{O(1)}$, then the Gap conjecture is true for residually finite groups.

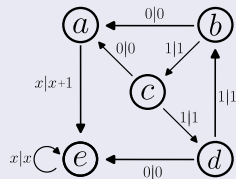
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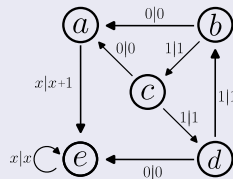


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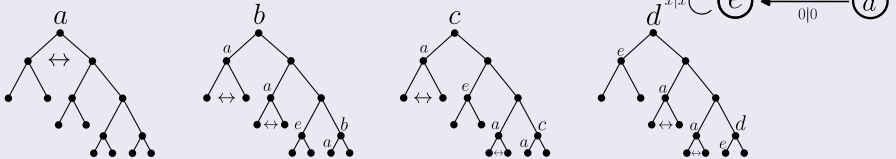
Writing $\sigma = (0, 1)$, the non-trivial states satisfy

$$a = \dot{\sigma} \cdot (e, e), \quad b = (a, c), \quad c = (a, d), \quad d = (e, b).$$



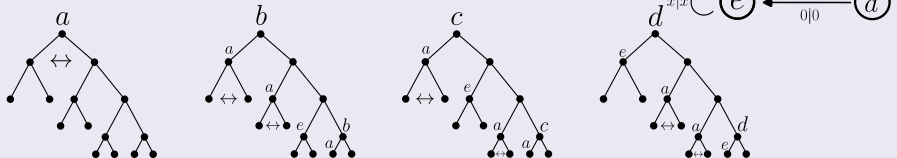
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Theorem (Grigorchuk, 1984)

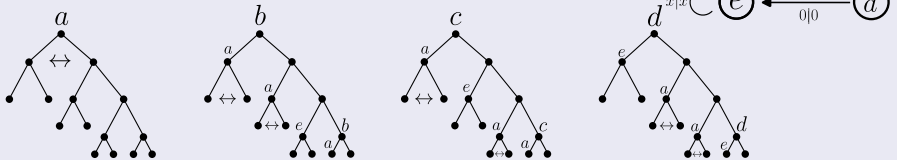
The growth function of $\mathcal{G}$ satisfies

$$e^{\sqrt{n}} \leq \tilde{\gamma}_{\mathcal{G}}^S(n) \leq e^{n^\alpha} \quad \text{for some } \frac{1}{2} < \alpha < 1,$$

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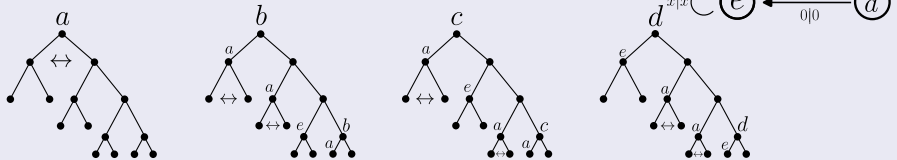
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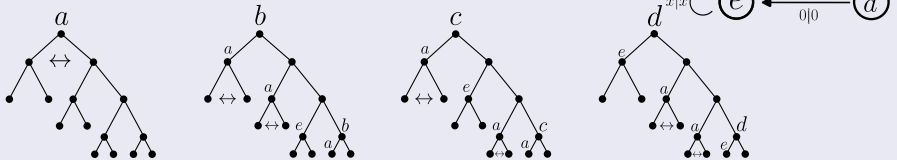
Remark

The states e, b, c, d form a Klein four-group:

$$\langle b, c, d \rangle = \{e, b, c, d\} \cong C_2 \times C_2, \quad b^2 = c^2 = d^2 = bcd = e.$$

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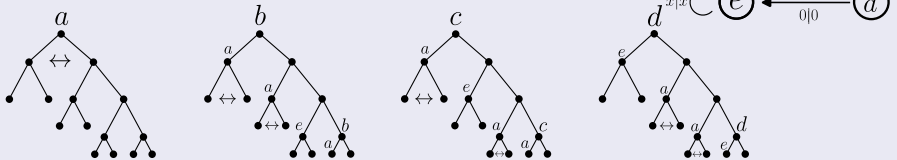
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- Discover the orbit of 111 by successively applying the generators.

●
111

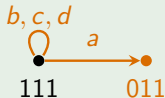
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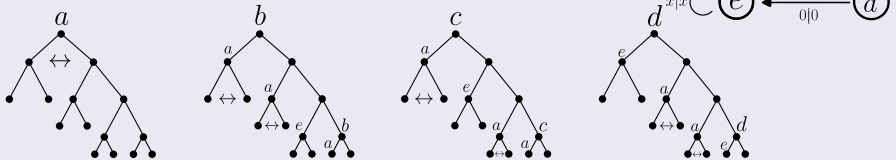
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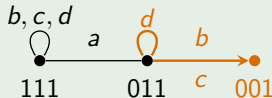
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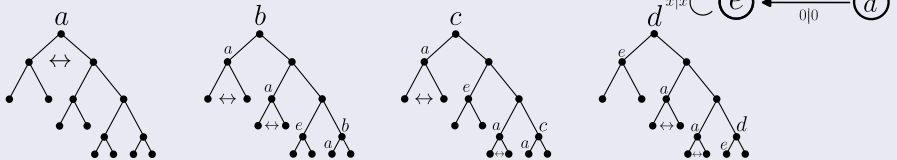
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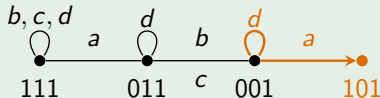
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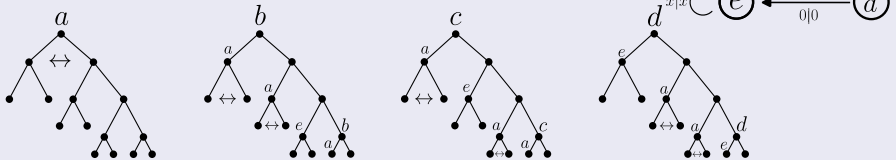
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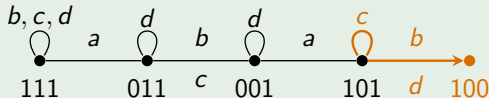
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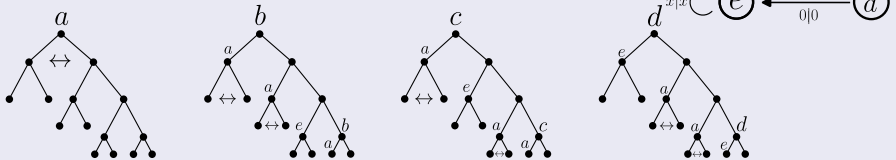
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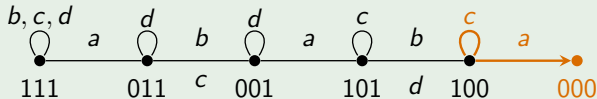
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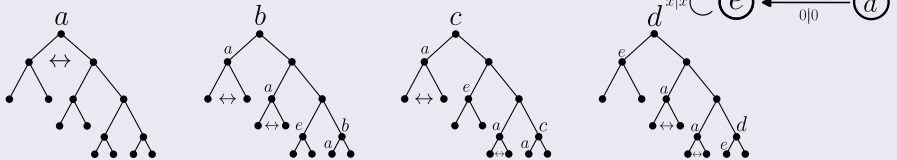
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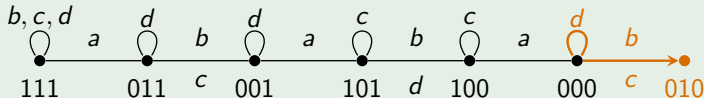
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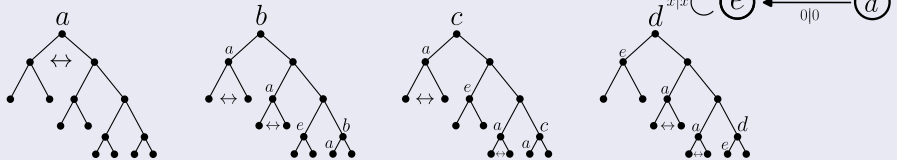
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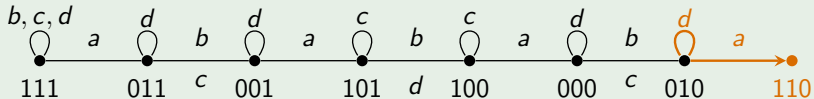
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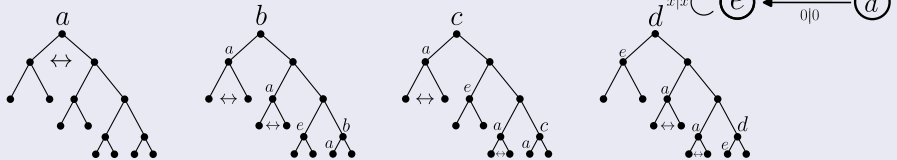
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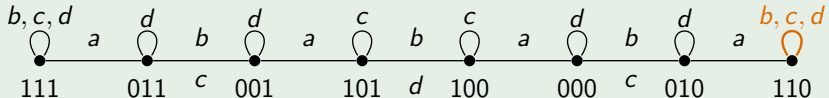
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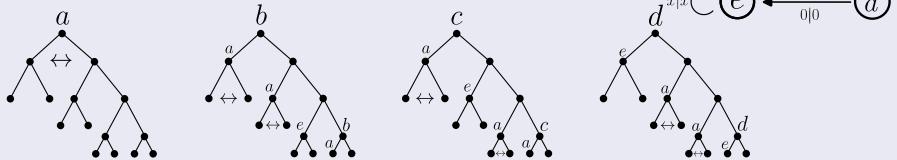
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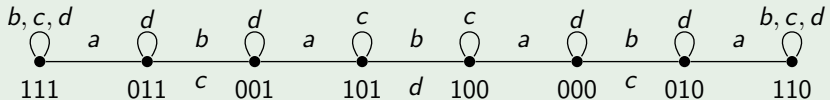
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- The graph is palindromic: reflection in its midpoint preserves all labels.

Definition (Schreier graph)

The *Schreier graph* $\text{Sch}(G, X, S)$ associated to an action $G \curvearrowright X$ and a generating set S is the graph with vertex set X such that for every $x \in X$ and $s \in S$ there is a directed edge $x \xrightarrow{s} xs$, labeled by s .

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For a word $w = s_1 s_2 \dots s_\ell \in S^*$ and a point $x \in X$, let

$$\text{Tr}_x(w) = \{ x, xs_1, xs_1 s_2, \dots, xs_1 \dots s_\ell \}$$

denote the trajectory from x to $xs_1 \dots s_\ell$ that is labeled by $s_1 \dots s_\ell$.

$$x \xrightarrow{s_1} xs_1 \xrightarrow{s_2} xs_1 s_2 \longrightarrow \dots \xrightarrow{s_\ell} xs_1 \dots s_\ell.$$

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Let $G \curvearrowright X$ be a transitive right action on an infinite set X and let S be a generating set of G . For every $x \in X$ and every $n \in \mathbb{N}$ we have

$$\sup_{|w|=n} |\text{Tr}_x(w)| = n + 1.$$

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With G, X, S, x, w as above, the inverted orbit of w at x is defined as

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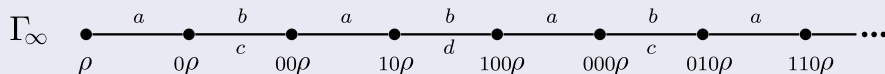
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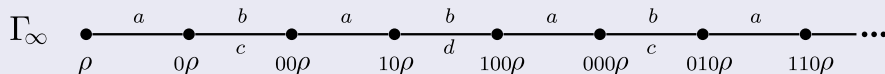
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Let $\rho = 1^\infty \in \mathfrak{C} = \partial\mathcal{T}$ and let Γ_∞ denote the Schreier graph of $\mathcal{G} \curvearrowright \{0,1\}^* \rho$ w.r.t. $S := \{a, b, c, d\}$.



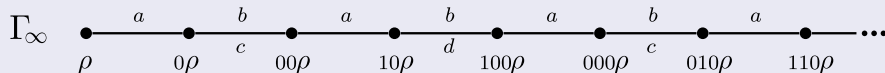
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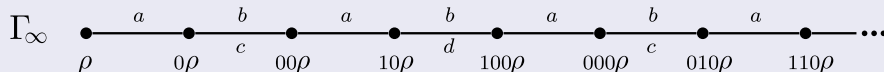
Theorem (Bartholdi, Erschler; 2012)

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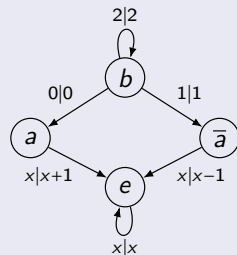
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Theorem (Bartholdi, Erschler; 2012)

The group $\text{Sym}_f(\{0,1\}^*\rho) \rtimes \mathcal{G}$ has intermediate growth. In particular, there are groups of intermediate growth that are not residually finite.

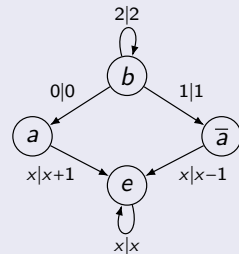
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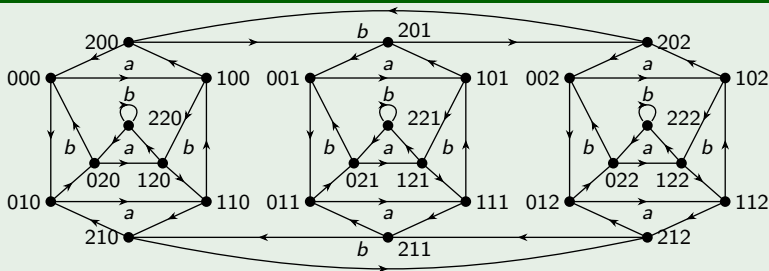


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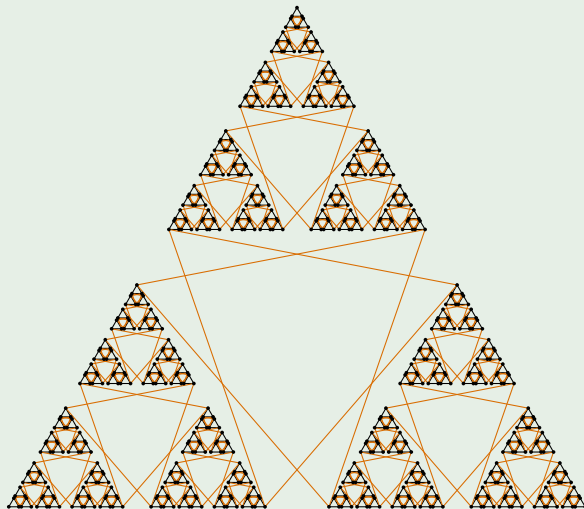


Example (Discovering the orbits of the Gupta–Sidki group)



Example (Discovering the orbit of 000000)

The orbit of 000000 under the Gupta–Sidki group $G = \langle a, b \rangle$ by successively applying a and b .



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Does there exist a finitely presented amenable group that is finitely presented but not elementary amenable?

Definition (Finite L -presentation)

A group G is said to have a *finite L -presentation* if there are finite sets S , Q , R and an endomorphism φ of the free group $F(S)$ such that

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The lamplighter group is not finitely presented; its standard presentation

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$$C_2 \wr \mathbb{Z} = \langle a, t \mid \varphi^i(a^2), \varphi^i([a, a^t]) \ (i \geq 0) \rangle.$$

Hence the lamplighter group has a finite L -presentation.

Theorem (Lysenok; 1985)

The Grigorchuk group is finitely L-presented:

$$\mathcal{G} = \langle a, c, d \mid \varphi^i(a^2), \varphi^i((ad)^4), \varphi^i((adacac)^4) \ (i \geq 0) \rangle,$$

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Theorem (Bartholdi; 2003)

*Every finitely generated, contracting, regular branch group is finitely L-presented.
This covers the Gupta–Sidki groups.*

Observation

Let $G = \langle S \mid \varphi^i(r), r \in R_0, i \geq 0 \rangle$ be finitely L -presented with φ injective on G , and put

$$\Gamma = \langle S, t \mid R_0, s^t = \varphi(s) (s \in S) \rangle.$$

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Proof.

- $\Gamma = (\bigcup_{i \geq 0} t^i \mathcal{G} t^{-i}) \rtimes \langle t \rangle$ contains $\mathcal{G}$, hence is not elementary amenable.
- $\mathcal{G}$ has subexponential growth, hence is amenable.
- Directed unions of amenable groups are amenable.
- Extensions of amenable groups by amenable groups are amenable.

Definition (Exponential growth rate)

The *exponential growth rate* of G with respect to S is defined as

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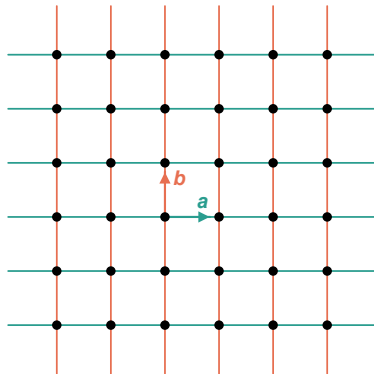
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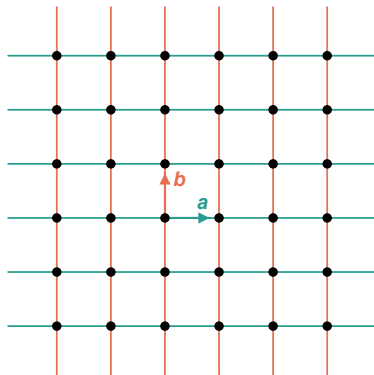
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- Collins, Edjvet, Gill (1994): $\omega(\langle a, b \mid aba^{-1} = b^3 \rangle, \{a, b\}) = 2$.

Whose Cayley graph is this?



A portion of a Cayley graph.

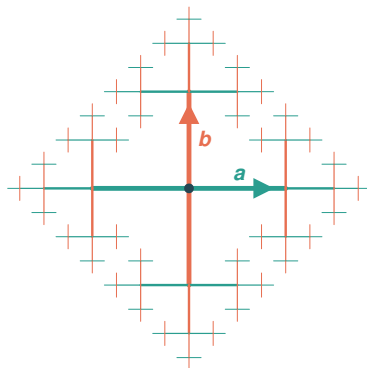
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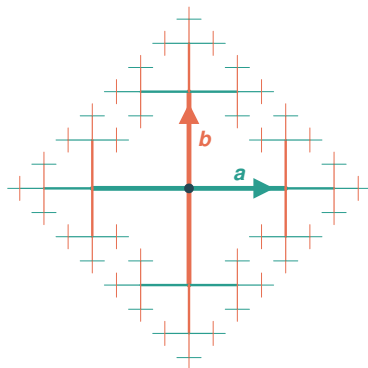
No, it is $\mathbb{Z}$ with respect to $a = 2584$ and $b = 4181$.

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It is $BS(2, 3) = \langle x, y \mid xy^2x^{-1} = y^3 \rangle$ with respect to $a = y$ and $b = xy^{100}$.

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Remark (How to prove uniform exponential growth)

G has uniform exponential growth if there is $R \in \mathbb{N}$ s.t. for every generating set S of G there are $g, h \in G$ with $|g|_S, |h|_S \leq R$ that generate a free monoid.

Facts (Positive results in non-positive curvature)

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Theorem (Breuillard, Varjú; 2020)

The following are equivalent:

- *For every d there is $\varepsilon(d) > 0$ such that $\omega(G) > 1 + \varepsilon(d)$ for every finitely generated non-virtually nilpotent $G \leq \mathrm{GL}_d(\overline{\mathbb{Q}})$.*
- *The Lehmer conjecture holds.*

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Idea

Groups of non-uniform exponential growth should appear with 'higher density' in the class of just-infinite groups.

Theorem (Wilson (1971) & Grigorchuk (2000))

Just-infinite groups fall into 3 classes:

- *Simple groups: Thompson groups, Burger-Mozes groups...*
- *Branch groups: Grigorchuk's group, Gupta-Sidki groups...*
- *Hereditarily just-infinite groups: $\mathbb{Z}$, D_∞ , $\mathrm{PSL}_n(\mathbb{Z})$ for $n \geq 3$, two exotic families.*

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Fact

All known examples of hereditarily just-infinite groups are either virtually nilpotent or of uniform exponential growth.

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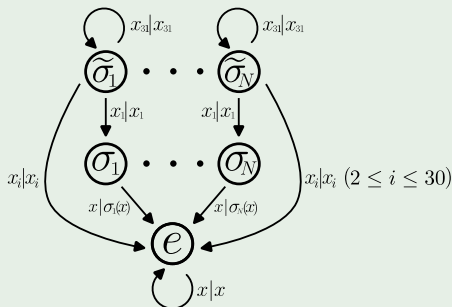
Every just-infinite quotient of a Wilson group is a branch group.

Fact

All known groups G with $G \cong G^{31} \rtimes \text{Alt}(31)$ are not finitely presented.

Reminder

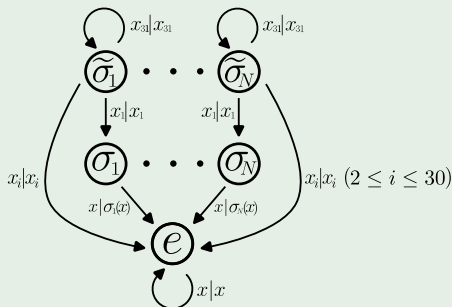
Segal's branch group Γ_{31} satisfies $\Gamma_{31} \cong \Gamma_{31} \wr \text{Alt}(31) = \Gamma_{31}^{31} \rtimes \text{Alt}(31)$.



The Moore diagram of Segal's branch group.

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Fact

Some of Segal's branch groups Γ_d contain free monoids of rank 2. Thus Wilson's result implies that Γ_d is of non-uniform exponential growth.

Question (Bartholdi; 2015)

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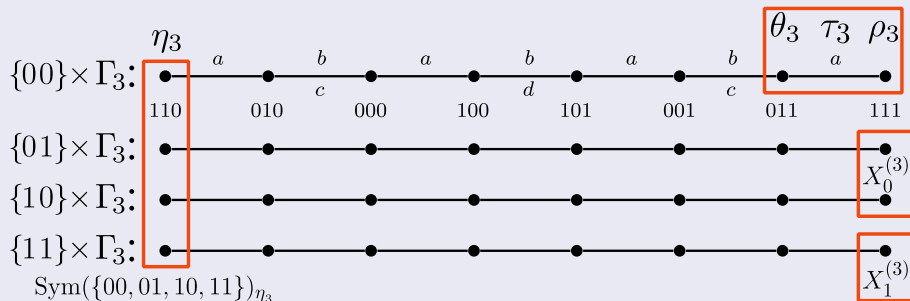
Theorem (Sauer, Schesler; 2026)

Each of these questions has an affirmative answer.

Theorem (Sauer, S.; 2026)

Thompson's group V is of non-uniform exponential growth.

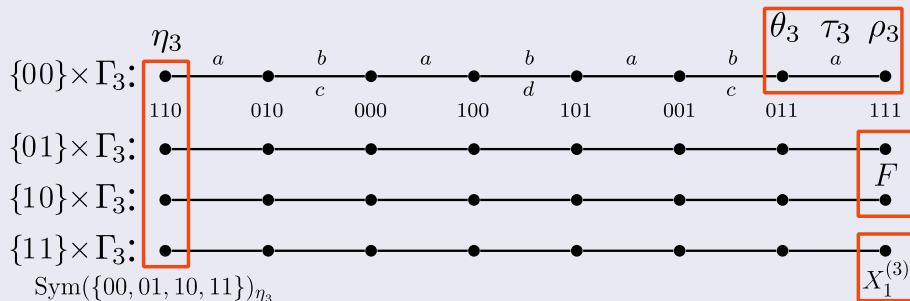
Proof idea.



Theorem (Sauer, S.; 2026)

Thompson's group V is of non-uniform exponential growth.

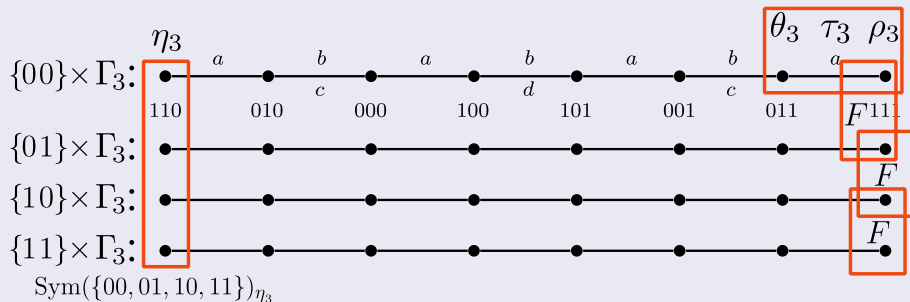
Proof idea.



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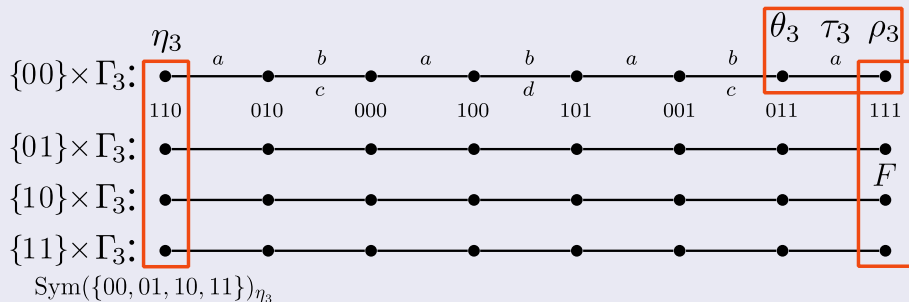
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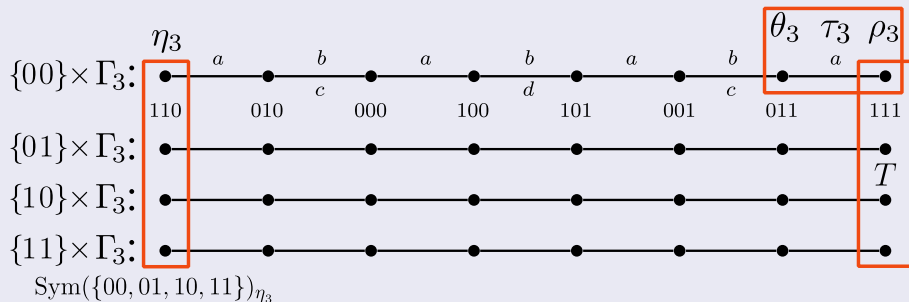
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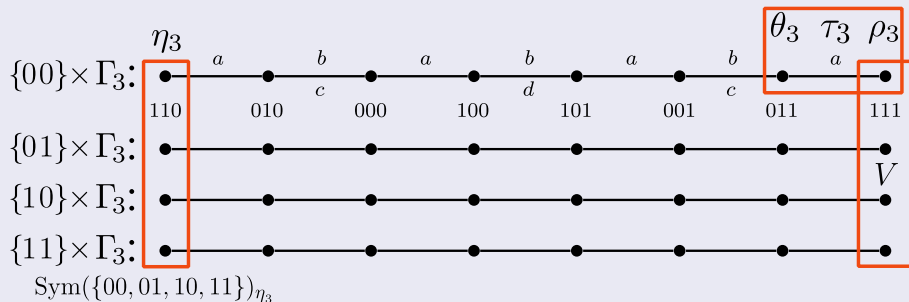
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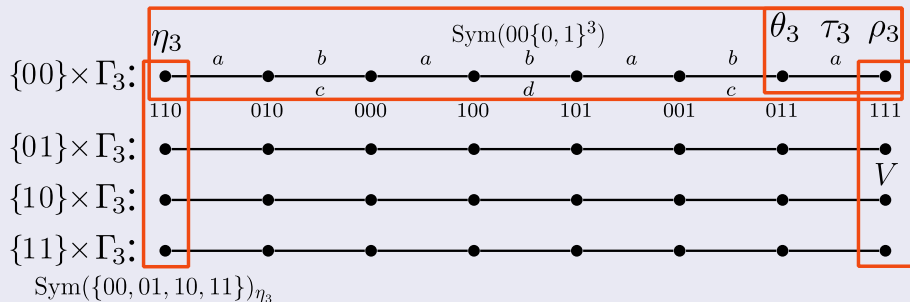
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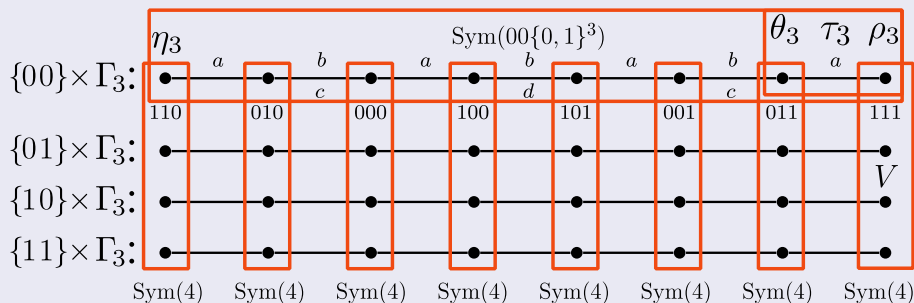
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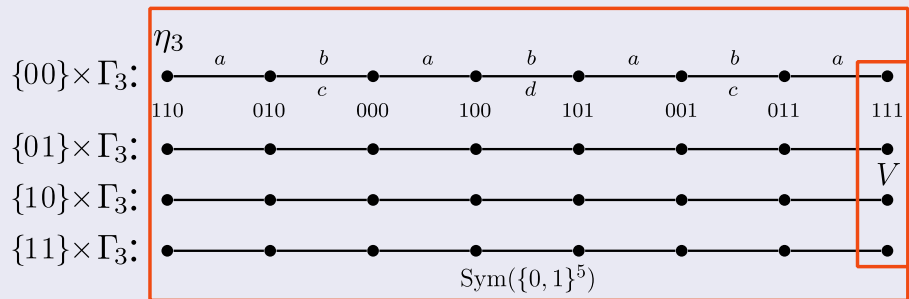
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