

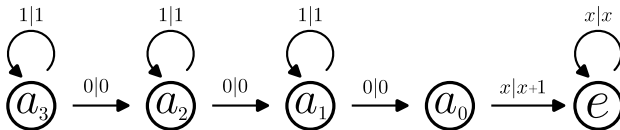
Mini course on automata and groups (part 4)

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18. September 2026

Talk given at the Shanghai Center for Mathematical Sciences (SCMS).



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- Writing $v = x_1 \dots x_\ell$, we obtain

$$g_v = (\dots((g_{x_1})_{x_2})_{x_3} \dots)_{x_\ell} \in G.$$

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- Writing $g = s_1 \dots s_\ell$ with $s_i \in S \cup S^{-1}$, these identities give $g_x \in G$ for every $x \in X$ and the first fact applies.

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Example

Automata groups are self-similar. Indeed, for $G(A) = \langle Q \rangle$ with $A = (X, Q, \pi, \lambda)$ we have $q_x = \pi(q, x) \in Q \subseteq G(A)$ for all $q \in Q$, $x \in X$. In fact, being an automata group is equivalent to being generated by a finite self-similar set.

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Questions

- *Zaremsky: Are braid groups self-similar?*
- *Is $\mathrm{SL}_n(\mathbb{Z})$ an automata group for $n \geq 3$?*

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- The natural action of $\text{SAff}_n(\mathbb{Z}_p)$ on $\mathbb{Z}_p^n$ induces an action on $T_{p,n}$.
- Goal: Show that the restricted action $\text{SAff}_n(\mathbb{Z}) \curvearrowright T_{p,n}$ is self-similar.

Proof continued.

- Let $g \in \text{SAff}_n(\mathbb{Z})$ be given by $g(v) = Av + b$, where $A \in \text{SL}_n(\mathbb{Z})$, $b \in \mathbb{Z}^n$.

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- From $Ax + b = x' + pb'$ it follows that $b' \in \mathbb{Z}^n$ and $g_x \in \text{SAff}_n(\mathbb{Z})$.
- $\Rightarrow \text{SAff}(\mathbb{Z})$ is self-similar.

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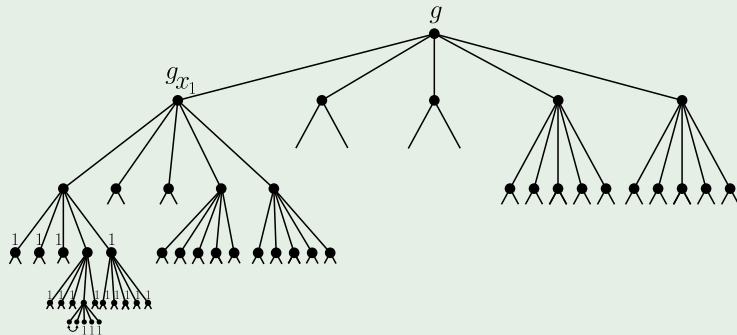
$$f: 2\mathbb{Z}^n \rtimes \mathrm{SL}_n(\mathbb{Z}) \rightarrow \mathbb{Z}^n \rtimes \mathrm{SL}_n(\mathbb{Z}), \quad (2b, A) \mapsto (b, A)$$

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Fact (Folklore)

Let $G \leq \text{Aut}(T_X)$ be a self-similar subgroup that acts transitively on X . For each $x \in X$, the map $\text{St}_G(x) \rightarrow G$, $g \mapsto g_x$ defines an essential virtual endomorphism of G .

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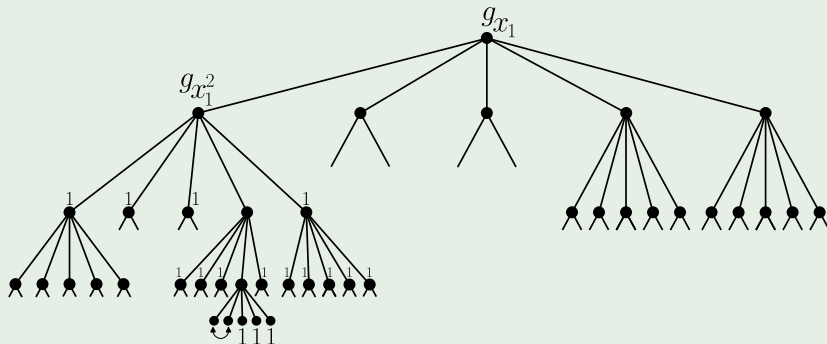


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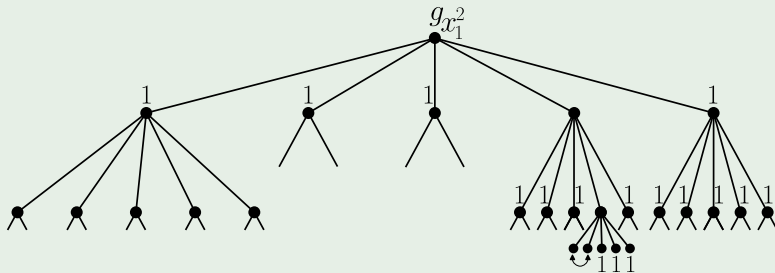


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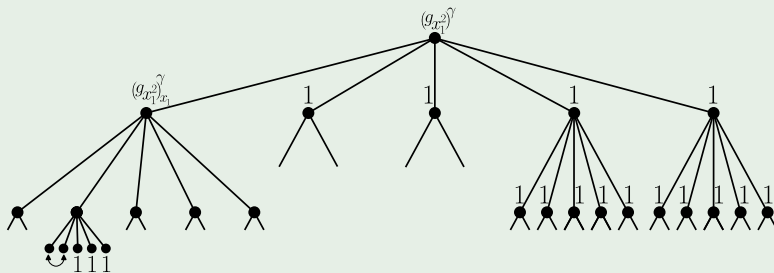


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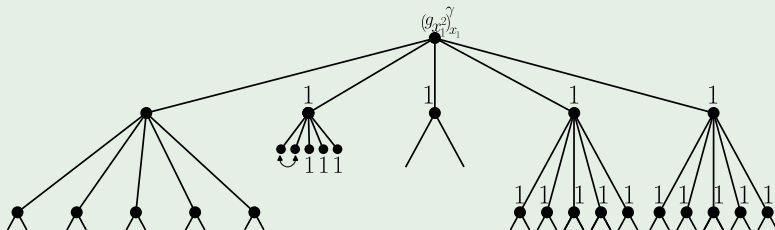


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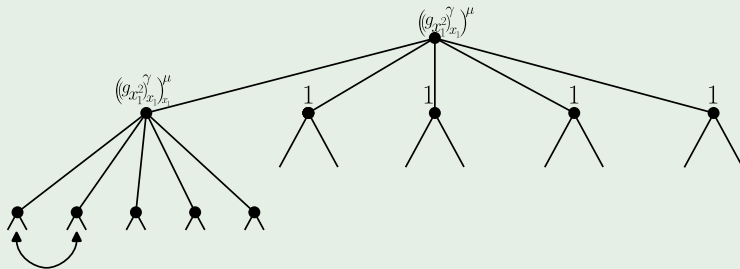


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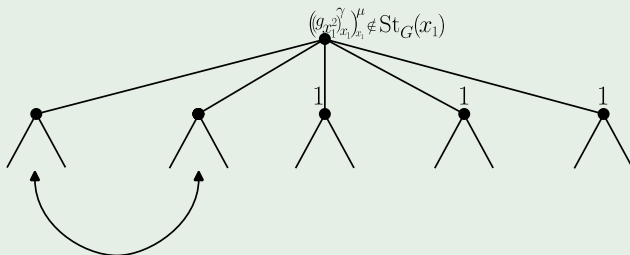


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- Let $\varphi: H \rightarrow G$ be a virtual endomorphism of G .
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If G admits an essential virtual endomorphism, then G admits a faithful self-similar action on a regular rooted tree that is transitive on the first level.

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- The map $\theta: G \rightarrow \text{Aut}(T_X)$ defines a self-similar action, which is faithful if and only if φ is essential.

Example

Consider the essential virtual endomorphism $\varphi: 2\mathbb{Z} \rightarrow \mathbb{Z}$, $2n \mapsto n$ and the transversal $X = \{0, 1\}$ of $\mathbb{Z}/2\mathbb{Z}$. The corresponding self-similar embedding $\theta: \mathbb{Z} \rightarrow \text{Aut}(T_X)$ satisfies

$$1 \mapsto \dot{\pi}_1 \cdot (\theta(\varphi(-\overline{1} + \overline{0} + 1 + 0)), \theta(\varphi(-\overline{1} + \overline{1} + 1 + 1))) = \sigma(\theta(0), \theta(1)),$$

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Fact

The following two properties of a group G are equivalent:

- *G admits a faithful self-similar action on a regular rooted tree.*
- *There exist virtual endomorphisms $f_i: H_i \rightarrow G$ with $i \in \{1, \dots, r\}$ such that there is no non-trivial normal subgroup $N \trianglelefteq G$ satisfying*

$$N \leq \bigcap_{i=1}^r H_i, \quad f_i(N) \leq N \quad (i = 1, \dots, r).$$

Example

The following groups are self-similar:

- finite groups,
- finitely generated abelian groups,
- free nilpotent groups $F_r/\gamma_{c+1}(F_r)$,
- free groups of finite rank,
- Grigorchuk's group,
- lamplighter groups $A \wr \mathbb{Z}$, where A is finite abelian,
- closed orientable surface groups $\pi_1(\Sigma_g)$,
- irreducible arithmetic lattices in linear semisimple Lie groups.

- $$\left(\left\{ \begin{pmatrix} 1 & b & c \\ 0 & M & a \\ 0 & 0 & 1 \end{pmatrix} : \begin{matrix} M \in E_{10}(R), \\ a, b^T \in R^{10}, c \in R \end{matrix} \right\} \rtimes_{\sigma} \mathbb{Z} \right) / \mathbb{Z}[\sqrt{2}],$$

$$R = \mathbb{Z}[\sqrt{2}, T, (1 + 2(1 + \sqrt{2})T)^{-1}], \quad \sigma(T) = (3 + 2\sqrt{2})T + 1, \quad \sigma(\sqrt{2}) = \sqrt{2}.$$

Here σ acts entry-wise and the quotient removes constant top-right entries.

Construction

Let $G \leq \text{Aut}(T_X)$ be a finitely generated self-similar group with finite symmetric generating set S . The action $G \curvearrowright T_X$ can be interpreted as an action of a finite *state asynchronous* automaton $A = (X, S, \pi, \lambda)$ where

- $\lambda: S \times X \rightarrow X, (s, x) \mapsto s(x),$
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Indeed, by extending the maps π, λ to $S^* \times X^*$ via

- $\lambda(vs, xw) = \lambda(v, \lambda(s, x)\pi(s, x)(w)),$
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Corollary

There are only countably many finitely generated self-similar groups.

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There are only countably many finitely generated self-similar groups.

Remark

Self-similar groups with unsolvable word problem cannot be recursively presented.

Example ($\text{SL}_2(\mathbb{Z})$)

Consider the generating set $S = \{a, a^{-1}, b, b^{-1}\}$ of $\text{SL}_2(\mathbb{Z})$, where

$$a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let $X = \{x_0, \dots, x_7\}$. Each entry below is $(\lambda(s, x), \pi(s, x))$, where $u = baaab^{-1}$ and $v = au$.

	a	a^{-1}	b	b^{-1}
x_0	(x_2, a)	(x_1, ε)	(x_3, ε)	(x_7, ε)
x_1	(x_0, ε)	(x_2, ε)	(x_6, v)	(x_2, v)
x_2	(x_1, ε)	(x_0, a^{-1})	(x_1, v^{-1})	(x_5, v^{-1})
x_3	(x_3, u)	(x_3, u^{-1})	(x_4, ε)	(x_0, ε)
x_4	(x_6, a)	(x_5, ε)	(x_7, ε)	(x_3, ε)
x_5	(x_4, ε)	(x_6, ε)	(x_2, v)	(x_6, v)
x_6	(x_5, ε)	(x_4, a^{-1})	(x_5, v^{-1})	(x_1, v^{-1})
x_7	(x_7, u)	(x_7, u^{-1})	(x_0, ε)	(x_4, ε)

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- $a(x_3x_2) = x_3u(x_2) = x_3(baaab^{-1})(x_2) = x_3(baaa)(x_5) = \dots = x_3(b)(x_5) = x_3x_2.$

Definition (Contracting)

A self-similar action $G \curvearrowright T_X$ is *contracting* if there exists a finite set $\mathcal{N} \subseteq G$ such that for every $g \in G$ there is some $k \in \mathbb{N}$ such that $g_v \in \mathcal{N}$ for every word $v \in X^*$ of length $> k$. The minimal set $\mathcal{N}$ with this property is called the *nucleus* of the action.

Lemma/Definition (Contraction constants)

Let $G \curvearrowright T_X$ be a self-similar action over a finite alphabet X , and let S be a finite generating set of G . Write $|g|_S$ for the word length of g with respect to S . The action is contracting if and only if there exist

$$k \geq 1, \quad 0 < \lambda < 1, \quad C \geq 0$$

such that

$$|g_v|_S \leq \lambda |g|_S + C \quad \text{for all } g \in G \text{ and } v \in X^k.$$

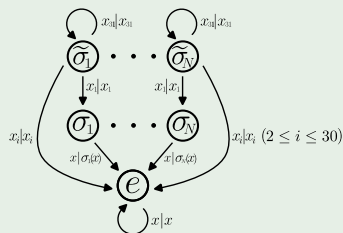
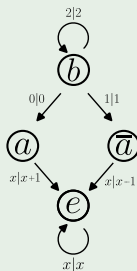
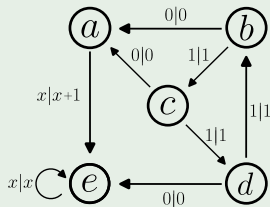
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Examples



The groups generated by the above automata are contracting.

Definition (Finitary/bounded/polynomial automorphisms)

For each $g \in \text{Aut}(T_X)$ and each $k \in \mathbb{N}$ let $\alpha_k(g) = |\{v \in X^k \mid g|_v \neq \text{id}\}|$. We say that g is

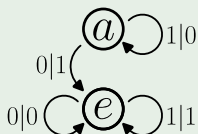
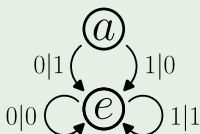
- *finitary* if $(\alpha_k(g))_k$ is eventually 0,
- *bounded* if $(\alpha_k(g))_k$ is bounded,
- *polynomial* if $(\alpha_k(g))_k$ is bounded above by a polynomial function.

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Example



The initial automaton A_a in the left resp. middle resp. right figure is finitary resp. bounded resp. exponential (not polynomial).

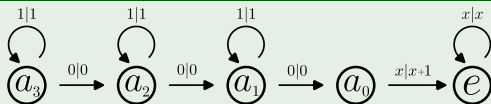
Proposition (Sidki; 2000)

For each finite state automorphism $g \in \text{Aut}(T_X)$ the sequence $(\alpha_k(g))_k$ either grows exponentially or polynomially.

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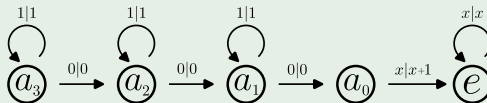


The Moore diagram of a finite state automorphism of $T_{\{0,1\}}$ of degree 2.

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Definition (B_n)

For each $n \in \mathbb{N}_0$ and each finite set X let $B_n(X)$ denote the group of all finite state automorphisms g of T_X , whose sequence $(\alpha_k(g))_k$ is bounded by a polynomial of degree n .

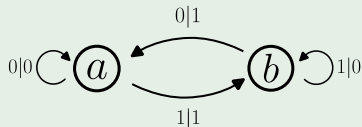
- $B_0(X)$ is called the group of bounded automata over X .
- $B_n(X)$ is called the group of polynomial automata of degree n over X .

Examples

- The groups of Gupta-Sidki and Segal lie in $B_0(X)$ for appropriate X .

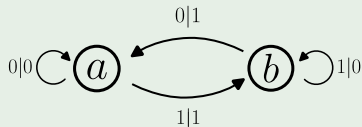
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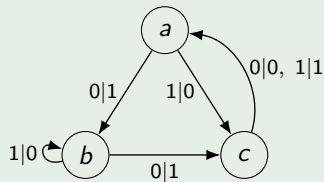


- Vorobets–Vorobets; 2007:

Let A be the automaton on the right. Then

$$G(A) = \langle A_a, A_b, A_c \rangle \cong F_3.$$

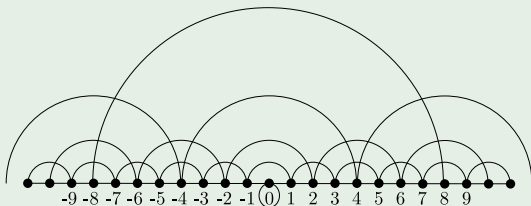
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Aleshin's automaton (1983).

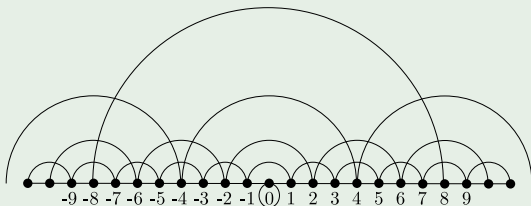
Example (The long range graph)

The *long range graph* Γ is the graph with vertex set $\mathbb{Z}$ and whose edges are given by $\bigcup_{k=0}^{\infty} E_k$, where $E_k = \{ (2^k(j - \frac{1}{2}), 2^k(j + \frac{1}{2})) \mid j \in \mathbb{Z} \}$.



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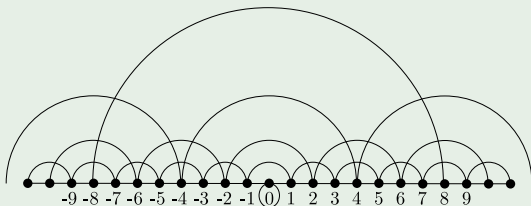


The *long range group* is the subgroup $\langle a, b \rangle \leq \text{Sym}(\mathbb{Z})$, where

- $a(x) = x + 2^k$, where $k = \min\{k \in \mathbb{N} \mid 2^k \nmid x\}$ for $x \neq 0$ and $a(0) = 0$,
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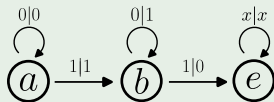
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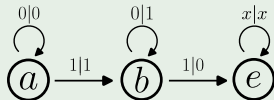


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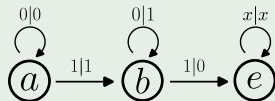


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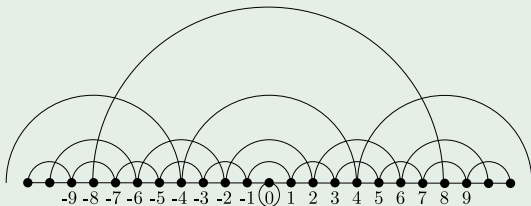
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Let $\xi = (0, 0, \dots) \in \partial T_X$. The Schreiergraph of the action $G(A) \curvearrowright G(A) \cdot \xi$ with respect to $\{a, b\}$ is isomorphic to the long range graph.



Theorem (Sidki; 2004)

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Theorem (Amir, Angel, Virág; 2013)

The group $B_1(X)$ is amenable for every finite set X .

Theorem (Juschenko, Nekrashevych, de la Salle; 2016)

A finitely generated subgroup $G \leq B_n(X)$ is amenable if its action on ∂T_X is recurrent.

Theorem (Amir, Angel, Virág; 2024)

The action of every finitely generated subgroup $G \leq B_2(X)$ on ∂T_X is recurrent.

Theorem (Amir, Virág; 2014)

For every finite set X with $|X| \geq 3$ the group $B_3(X)$ contains a finitely generated subgroup whose action on ∂T_X is transient.

Theorem (Bondarenko; 2011)

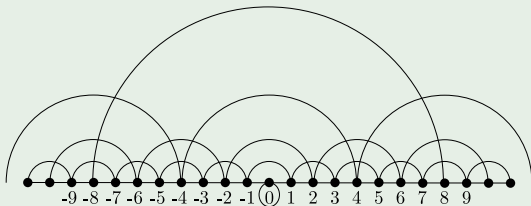
Let $G \leq B_m(X)$ be a finitely generated group with finite generating set S . There exists an $A > 0$ such that for every $\xi \in \partial T_X$ the growth of the Schreier graph $\Gamma(G, S, G \cdot \xi)$ is bounded above by $n \mapsto A^{\log(n)^{m+1}}$.

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Example

The long-range graph has quasi-polynomial growth.

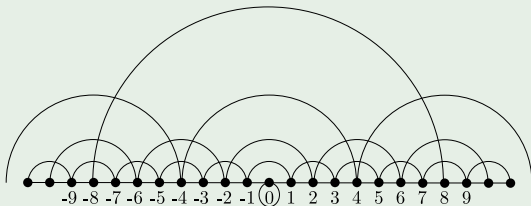


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Example

The long-range graph has quasi-polynomial growth.



Corollary

A subgroup $G \leq B_n(X)$ does not have property (T) unless G is finite.

Theorem (Bondarenko, Nekrashevich; 2008)

Every finitely generated self-similar subgroup of $B_0(X)$ is contracting.

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Conjecture (Nekrashevich; 2005)

Contracting groups are amenable.

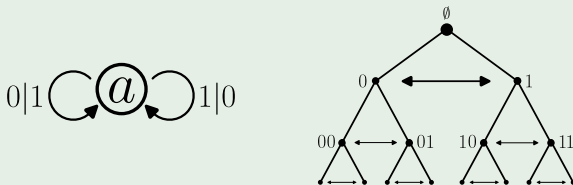
Definition (Contracting)

A self-similar action $G \curvearrowright T_X$ is *contracting* if there exists a finite set $\mathcal{N} \subseteq G$ such that for every $g \in G$ there is some $k \in \mathbb{N}$ such that $g_v \in \mathcal{N}$ for every word $v \in X^*$ of length $> k$. The minimal set $\mathcal{N}$ with this property is called the *nucleus* of the action.

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Example



Let A be the automaton over the alphabet $X = \{0, 1\}$ that is given by the Moore-diagram above. The group $G(A)$ is a self-similar contracting group that does not lie in $B_d(X)$ for any $d \in \mathbb{N}$.

Reminder

$\mathcal{G} = \langle a, b, c, d \rangle$, where $\sigma = (0, 1)$, $a = \sigma \cdot (e, e)$, $b = (a, c)$, $c = (a, d)$, $d = (e, b)$.

Theorem (Bartholdi; 1998)

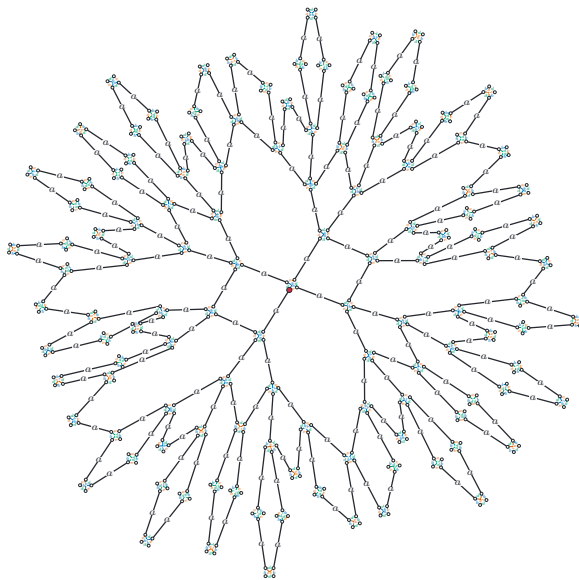
For the standard generating set $S := \{a, b, c, d\}$, let

$$\gamma_{\mathcal{G}}^S(n) := \#\{g \in \mathcal{G} \mid |g|_S \leq n\}.$$

Then

$$\exp(\sqrt{n}) \preceq \gamma_{\mathcal{G}}^S(n) \preceq \exp(n^\alpha), \quad \alpha = \frac{\log 2}{\log 2 - \log \eta},$$

where $\eta \in (0, 1)$ is the positive root of $\eta^3 + \eta^2 + \eta - 2 = 0$.

**Generators**

a: between cells

b, c, d: inside each

 K_4 **one K_4 cell**

Geometry of the reduced $B_{10}(e)$ drawing; K_4 cells restored, boundary leaves omitted

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$$(ad)^2 = \dot{\sigma} \cdot (e, b) \cdot \dot{\sigma} \cdot (e, b) = (e, b)^{\dot{\sigma}} \cdot (e, b) = (b, e) \cdot (e, b) = (b, b).$$

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- $\Rightarrow (ad)^4 = ((ad)^2)^2 = (b, b)^2 = (e, e) = e$.

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Corollary

Let $B = \langle\langle b \rangle\rangle_{\mathcal{G}}$. Then $\mathcal{G}/B$ is generated by a, d and satisfies $a^2 = d^2 = (ad)^4 = e$. Hence $\mathcal{G}/B$ is a quotient of D_4 and $|\mathcal{G}/B| = [\mathcal{G} : B] \leq 8$.

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Proof.

In $\mathcal{G}/B$ we have $e = bcd = cd$. Thus $c = c^{-1} = d$ and

$$\mathcal{G}/B = \langle a, b, c, d \rangle = \langle a, d \rangle.$$

Definition (permutational wreath product)

Let G be a group that acts on a set X and let H be a group. The permutational wreath product $G \wr_X H$ of G and H with respect to $G \curvearrowright X$ is defined as

$$G \wr_X H = G \ltimes \bigoplus_{x \in X} H.$$

Fact

Consider $G \wr_X H$, where X is finite and let $\varphi: H \rightarrow A$ be a homomorphism where A is abelian. The following map is a homomorphism

$$\widehat{\varphi}: \Gamma \longrightarrow A, \quad (g, (h_x)_{x \in X}) \longmapsto \sum_{x \in X} \varphi(h_x).$$

Proof.

$$\begin{aligned} \widehat{\varphi}(g(h_x)g'(h'_x)) &= \widehat{\varphi}(gg'g'^{-1}(h_x)g'(h'_x)) = \widehat{\varphi}(gg'(h_{g'x}h'_x)) = \sum_{x \in X} \varphi(h_{g'x}h'_x) \\ &= \sum_{x \in X} \varphi(h_{g'x}) + \sum_{x \in X} \varphi(h'_x) = \widehat{\varphi}(g(h_x)) + \widehat{\varphi}(g'(h'_x)). \end{aligned}$$

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Example

Let $G \leq \text{Aut}(T_X)$ be self-similar and let $\varphi: G \rightarrow A$ be a morphism to an abelian group A . For each $n \in \mathbb{N}_0$, the following map defines a homomorphism

$$\widehat{\varphi}_n: G \longrightarrow A, \quad g \longmapsto \sum_{v \in X^n} \varphi(g_v).$$

Example

Let $\mathcal{G} \leq \text{Aut}(T_X)$ be the Grigorchuk group. The map

$$\chi: \mathcal{G} \longrightarrow \text{Sym}(X)^3 \cong \mathbb{F}_2^3, \quad g \longmapsto (\sigma_g, \sigma_{g_0} \sigma_{g_1}, \sigma_{g_{00}} \sigma_{g_{01}} \sigma_{g_{10}} \sigma_{g_{11}})$$

is an epimorphism with kernel $[\mathcal{G}, \mathcal{G}]$. Thus $\mathcal{G}^{\text{ab}} \cong \mathbb{F}_2^3$.

Reminder

$\mathcal{G} = \langle a, b, c, d \rangle$, where $\sigma = (0, 1)$, $a = \dot{\sigma} \cdot (e, e)$, $b = (a, c)$, $c = (a, d)$, $d = (e, b)$.

Lemma

The element $(ad)^2$ does not lie in $B = \langle\langle b \rangle\rangle_{\mathcal{G}}$.

Proof.

- The map $\chi: \text{St}_{\mathcal{G}}(1) \rightarrow \text{Sym}(X)$, $(g_0, g_1) \mapsto \sigma_{g_0} \cdot \sigma_{(g_{10})} \sigma_{(g_{11})}$ is a homomorphism.
- Let $g = \sigma_g(g_0, g_1) \in \mathcal{G}$. If $\sigma_g = \text{id}$, then $g \in \text{St}_{\mathcal{G}}(1)$ and we obtain

$$\chi(b^g) = \chi(b) = \sigma_{b_0} \sigma_{b_{10}} \sigma_{b_{11}} = \sigma_a \sigma_a \sigma_d = e.$$

If $\sigma_g \neq \text{id}$, then

$$\chi(b^g) = \chi((a, c)^{\sigma_g(g_0, g_1)}) = \chi((c, a)^{(g_0, g_1)}) = \chi((c, a)) = \sigma_c \cdot \sigma_{a_0} \sigma_{a_1} = \text{id}.$$

- Thus B is contained in the kernel of χ .
- $\chi((ad)^2) = \chi(d^a d) = \chi((b, e)(e, b)) = \chi((b, b)) = \sigma_b \sigma_{b_0} \sigma_{b_1} = \text{id} \cdot \sigma_a \cdot \text{id} \neq \text{id}$.
- Consequently $(ad)^2 \notin B$.

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- The generators of $\mathcal{G}$ satisfy $a^2 = b^2 = c^2 = d^2 = bcd = (ad)^4 = e$.
- Let $B = \langle\langle b \rangle\rangle_{\mathcal{G}}$. Then $\mathcal{G}/B$ is generated by $\bar{a}, \bar{d}$ and we have $\bar{c} = \bar{d}$.
- The element $(ad)^2$ does not lie in $B = \langle\langle b \rangle\rangle_{\mathcal{G}}$.

Corollary

There is an epimorphism

$$\theta: \mathcal{G} \twoheadrightarrow D := \langle a, d \rangle \cong D_4 \quad \text{with} \quad a \mapsto d, \quad b \mapsto 1, \quad c \mapsto a, \quad d \mapsto a.$$

Moreover we have $\ker(\theta) = B$ and $\theta^2(t) = t$ ($t \in D$).

Proof.

- The map $D_4 := \langle s, t \mid s^2, t^2, (st)^4 \rangle \rightarrow \langle a, d \rangle$, $s \mapsto a, t \mapsto d$ is an isomorphism.
- The canonical map $\langle a, d \rangle \rightarrow \mathcal{G}/B$ defines an isomorphism.
- The map $D_4 \rightarrow D$, $s \mapsto t, t \mapsto s$ defines an isomorphism.
- $\theta: \mathcal{G} \xrightarrow{y \mapsto \bar{y}} \mathcal{G}/B \xrightarrow{\bar{a} \mapsto a, \bar{d} \mapsto d} D \xrightarrow{a \mapsto s, d \mapsto t} D_4 \xrightarrow{s \mapsto t, t \mapsto s} D_4 \xrightarrow{s \mapsto a, t \mapsto d} D$

Lemma

There is an injective endomorphism $\sigma: \mathcal{G} \rightarrow \mathcal{G}$ with

$$\sigma(a) = c^a, \quad \sigma(b) = d, \quad \sigma(c) = b, \quad \sigma(d) = c,$$

and it satisfies $\sigma(g) = (\theta(g), g)$ for every $g \in \mathcal{G}$.

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Proof.

- Let $q: F_4 = F(A, B, C, D) \rightarrow \mathcal{G}$, $A \mapsto a, B \mapsto b, C \mapsto c, D \mapsto d$.
- Let $\tilde{\sigma}: F_4 \rightarrow \text{St}_{\mathcal{G}}(1) \subseteq \mathcal{G} \times \mathcal{G}$ be given by $A \mapsto c^a, B \mapsto d, C \mapsto b, D \mapsto c$.
- $\tilde{\sigma}(A) = c^a = (d, a) = (\theta(a), a) = (\theta(q(A)), q(A))$,
- $\tilde{\sigma}(B) = d = (e, b) = (\theta(b), b) = (\theta(q(B)), q(B))$,
- $\tilde{\sigma}(C) = b = (a, c) = (\theta(c), c) = (\theta(q(C)), q(C))$,
- $\tilde{\sigma}(D) = c = (a, d) = (\theta(d), d) = (\theta(q(D)), q(D))$.
- $\tilde{\sigma}(w) = (\theta(q(w)), q(w))$ for every $w \in F_4$.
- If $w \in \ker(q)$, then $\tilde{\sigma}(w) = (e, e)$. Hence $\ker(q) \leq \ker(\tilde{\sigma})$.
- $\tilde{\sigma}$ induces a well-defined endomorphism $\sigma: \mathcal{G} \rightarrow \mathcal{G}$.
- From $\sigma(g) = (\theta(g), g)$ for each $g \in \mathcal{G}$ it follows that σ is injective.

Reminder

$\mathcal{G} = \langle a, b, c, d \rangle$, where $\sigma = (0, 1)$, $a = \dot{\sigma} \cdot (e, e)$, $b = (a, c)$, $c = (a, d)$, $d = (e, b)$.
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Lemma

The word metric on $\mathcal{G}$ with respect to $S = \{a, b, c, d\}$ satisfies

$$|\sigma(g)|_S \leq 2|g|_S + 1 \quad (g \in \mathcal{G}).$$

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Proof.

- $a^2 = 1$ and $\{1, b, c, d\}$ is a Klein four-group.
- Every minimal representative w of an element $g \in \mathcal{G}$ is of the form $s_0 a s_1 a \dots s_n a s_{n+1}$ with $s_0, s_{n+1} \in \{e, b, c, d\}$ and $s_i \in \{b, c, d\}$ for $1 \leq i \leq n$.
- Let A and M be the numbers of a -letters, resp. $\{b, c, d\}$ -letters, in w . Then

$$|g|_S = A + M, \quad A \leq M + 1.$$

- We have $|\sigma(a)| = 3$ and $|\sigma(y)| = 1$ for $y \in \{b, c, d\}$. Hence

$$|\sigma(g)|_S \leq 3A + M = 2(A + M) + (A - M) \leq 2|g|_S + 1.$$

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- The word metric on $\mathcal{G}$ w.r.t $S = \{a, b, c, d\}$ satisfies $|\sigma(g)|_S \leq 2|g|_S + 1$.

Lemma

The map

$$F: \mathcal{G} \times \mathcal{G} \longrightarrow \text{St}_{\mathcal{G}} \subseteq \mathcal{G} \times \mathcal{G}, \quad (g_0, g_1) \mapsto \sigma(g_0)^a \sigma(g_1)$$

satisfies

$$F(g_0, g_1) = (g_0 \theta(g_1), \theta(g_0) g_1) \quad \text{for all } g_0, g_1 \in \mathcal{G}.$$

Moreover

$$|F(g_0, g_1)|_S \leq 2|g_0|_S + 2|g_1|_S + 4.$$

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Proof.

- $F(g_0, g_1) = (\theta(g_0), g_0)^a (\theta(g_1), g_1) = (g_0 \theta(g_1), \theta(g_0) g_1)$.
- $|F(g_0, g_1)|_S \leq |\sigma(g_0)|_S + 2 + |\sigma(g_1)|_S \leq 2|g_0|_S + 1 + 2 + 2|g_1|_S + 1$.

Reminder

- $F: \mathcal{G} \times \mathcal{G} \longrightarrow \text{St}_{\mathcal{G}} \subseteq \mathcal{G} \times \mathcal{G}, (g_0, g_1) \mapsto \sigma(g_0)^a \sigma(g_1)$
- $F(g_0, g_1) = (g_0 \theta(g_1), \theta(g_0) g_1)$ for all $g_0, g_1 \in \mathcal{G}$.

Lemma

Every non-empty fibre of F contains exactly 8 elements.

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Proof.

- Let $(g_0, g_1) \in \mathcal{G} \times \mathcal{G}$, let $(h_0, h_1) \in F^{-1}(g_0, g_1)$, and let $t := g_0^{-1} h_0$.
- $g_0 \theta(g_1) = F(g_0, g_1)_0 = F(h_0, h_1)_0 = h_0 \theta(h_1) = g_0 t \theta(h_1)$.
- $\theta(g_0) g_1 = F(g_0, g_1)_1 = F(h_0, h_1)_1 = \theta(h_0) h_1 = \theta(g_0) \theta(t) h_1$.
- $\Rightarrow t = \theta(g_1) \theta(h_1)^{-1} \in D$ and $h_1 = \theta(t)^{-1} g_1$.
- $\Rightarrow (h_0, h_1) = (g_0 t, \theta(t)^{-1} g_1)$ with $t \in D \Rightarrow |F^{-1}((g_0, g_1))| \leq |D| = 8$.
- Consider now $(h_0, h_1) = (g_0 t, \theta(t)^{-1} g_1)$ for some $t \in D$.
- $F(h_0, h_1)_0 = g_0 t \theta(\theta(t)^{-1} g_1) = g_0 t \theta^2(t)^{-1} \theta(g_1) = g_0 \theta(g_1) = F(g_0, g_1)_0$.
- $F(h_0, h_1)_1 = \theta(g_0 t) \theta(t)^{-1} g_1 = \theta(g_0) g_1 = F(g_0, g_1)_1$.
- $(h_0, h_1) \in F^{-1}(g_0, g_1)$.

Reminder

- $F: \mathcal{G} \times \mathcal{G} \longrightarrow \text{St}_{\mathcal{G}} \subseteq \mathcal{G} \times \mathcal{G}, (g_0, g_1) \mapsto \sigma(g_0)^a \sigma(g_1)$
- $|F(g_0, g_1)|_S \leq 2|g_0|_S + 2|g_1|_S + 4$
- Every non-empty fibre of F contains exactly 8 elements.
- We write $S = \{a, b, c, d\} \subseteq \mathcal{G}, B_S(n) = \{g \in \mathcal{G} \mid |g|_S \leq n\}, \gamma_{\mathcal{G}}^S(n) = |B_S(n)|$.

Lemma

For each $n \in \mathbb{N}$ we have $\gamma_{\mathcal{G}}^S(4n-2) \geq \frac{\gamma_{\mathcal{G}}^S(n-2)^2}{8}$.

Proof of the lower bound.

- Pigeonhole principle:

$$|F(B_S(n) \times B_S(n))| \geq \frac{1}{8} |B_S(n) \times B_S(n)| = \frac{\gamma_{\mathcal{G}}^S(n)^2}{8}.$$

- $F(B_S(n) \times B_S(n)) \subseteq B_S(4n+4) \implies \gamma_{\mathcal{G}}^S(4n+4) \geq \frac{\gamma_{\mathcal{G}}^S(n)^2}{8}.$
- $\gamma_{\mathcal{G}}^S(4n-2) \geq \gamma_{\mathcal{G}}^S(4n-4) \geq \frac{\gamma_{\mathcal{G}}^S(n-2)^2}{8}.$

Reminder

For each $n \in \mathbb{N}$ we have $\gamma_G^S(4n-2) \geq \frac{\gamma_G^S(n-2)^2}{8}$.

Lemma

For every $k \geq 0$ and every $n \geq 2$,

$$\gamma_G^S(4^k n - 2) \geq 8 \left(\frac{\gamma_G^S(n-2)}{8} \right)^{2^k}.$$

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Proof.

- For $k = 0$ the claim is clear.
- Assume the lemma holds for k . Replaced n in the remark by $4^k n$ gives us

$$\gamma_{\mathcal{G}}^S(4^{k+1}n - 2) \geq \frac{\gamma_{\mathcal{G}}^S(4^k n - 2)^2}{8} \geq \frac{1}{8} \left[8 \left(\frac{\gamma_{\mathcal{G}}^S(n-2)}{8} \right)^{2^k} \right]^2 = 8 \left(\frac{\gamma_{\mathcal{G}}^S(n-2)}{8} \right)^{2^{k+1}}.$$

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Theorem (Grigorchuk; 1984)

The growth function of Grigorchuk's group $\mathcal{G}$ satisfies $\gamma_{\mathcal{G}}^S(n) \succeq e^{\sqrt{n}}$.

Proof.

- Choose $n_0 \geq 2$ such that $q := \frac{\gamma_{\mathcal{G}}^S(n_0-2)}{8} > 1$.
- For $N_k := 4^k n_0 - 2$, the above remark gives us $\gamma_{\mathcal{G}}^S(N_k) \geq 8q^{2^k}$.
- For sufficiently large N , choose k with $N_k \leq N < N_{k+1} = 4^{k+1} n_0 - 2$. Then

$$\frac{N}{n_0} < \frac{N+2}{n_0} < 4^{k+1} \implies \frac{\sqrt{N}}{2\sqrt{n_0}} < 2^k.$$

- $\gamma_{\mathcal{G}}^S(N) \geq \gamma_{\mathcal{G}}^S(N_k) \geq 8q^{2^k} = \exp(\log(8q^{2^k})) \geq \exp(\log(q)2^k) \geq \exp\left(\log(q)\frac{\sqrt{N}}{2\sqrt{n_0}}\right)$
- By setting $c := \frac{\log(q)}{2\sqrt{n_0}} > 0$ we obtain $\gamma_{\mathcal{G}}^S(N) \geq \exp(c\sqrt{N})$ for large enough N .

Theorem (Bartholdi, 1998)

Let $S := \{a, b, c, d\} \subseteq \mathcal{G}$. There is a constant $C > 0$ such that for each $n \in \mathbb{N}$

$$\gamma_{\mathcal{G}}^S(n) \leq \exp(Cn^\alpha), \quad \alpha := \frac{\log 2}{\log 2 - \log \eta},$$

where η is the positive root of $\eta^3 + \eta^2 + \eta - 2 = 0$.

Definition

For the root $\eta \in (0, 1)$ of $\eta^3 + \eta^2 + \eta - 2 = 0$, assign the weights

$$\|a\| = 1 - \eta^3, \quad \|b\| = \eta^3, \quad \|c\| = 1 - \eta^2, \quad \|d\| = 1 - \eta.$$

For $g \in \mathcal{G}$, let $\|g\|$ be the minimum total weight of a word representing g .

Lemma

If $g = \dot{\sigma}_g \cdot (g_0, g_1)$, then $\|g_0\| + \|g_1\| \leq \eta(\|g\| + \|a\|)$.

Lemma

By changing $\|\cdot\|$ by a bounded amount, we obtain for large $\|g\|$ that

$$\|g_0\| + \|g_1\| \leq \eta\|g\|.$$

Construction

Fix K so large that

$$\|g_0\| + \|g_1\| \leq \eta \|g\| \quad \text{whenever } \|g\| > K.$$

We associate to every $g \in \mathcal{G}$ a finite rooted tree $R(g)$.

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Lemma

The map $g \mapsto R(g)$ is injective.

Notation

Let $\#R(g)$ denote the number of leaves of $R(g)$, and put

$$\alpha := \frac{\log 2}{\log 2 - \log \eta} \in (0, 1).$$

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Lemma

There is a constant $E > 1$ such that the number of possible labelled trees $R(g)$ with at most m leaves is at most E^m .

Theorem (Bartholdi, 1998)

Let $S := \{a, b, c, d\} \subseteq \mathcal{G}$. There is a constant $C > 0$ such that for each $n \in \mathbb{N}$

$$\gamma_{\mathcal{G}}^S(n) \leq \exp(Cn^\alpha), \quad \alpha := \frac{\log 2}{\log 2 - \log \eta},$$

where η is the positive root of $\eta^3 + \eta^2 + \eta - 2 = 0$.

Proof sketch.

By setting $M := \max\{\|a\|, \|b\|, \|c\|, \|d\|\}$ we obtain

$$\|g\| = \max\{1, \|g\|_0 - C\} \leq 1 + M \cdot |g|_S.$$

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