

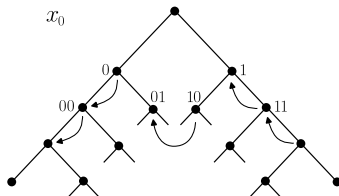
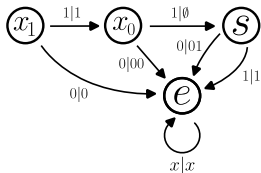
Mini course on automata and groups (part 5)

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18. September 2026

Talk given at the Shanghai Center for Mathematical Sciences (SCMS).



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Construction

The maps π, λ can be extended on $Q \times X^*$ by setting

- $\pi(q, xw) = \pi(\pi(q, x), w)$,
- $\lambda(q, xw) = \lambda(q, x)\lambda(\pi(q, x), w)$,

for $x \in X$, $q \in Q$, and $w \in X^*$.

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for $x \in X$, $q \in Q$, and $w \in X^*$. Similarly, λ can be defined on $Q \times X^\omega$.

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An asynchronous automaton $A = (X, Q, \pi, \lambda)$ with a fixed state q is called *initial* and is denoted by A_q . We identify A_q with $\lambda(q, \cdot)$ and with q .

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The *Moore diagram* of an asynchronous automaton $A = (X, Q, \pi, \lambda)$ is the labeled graph with vertex set Q and edge set $\{ (q, \pi(q, x)) \mid q \in Q, x \in X \}$, where $(q, \pi(q, x))$ is labelled by $x|\lambda(q, x)$.

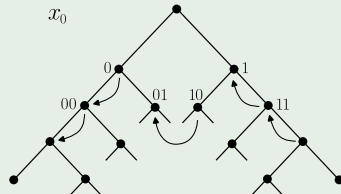
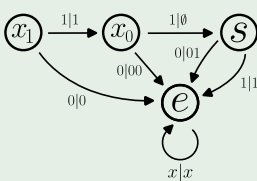
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Example



The subgroup $\langle x_1, x_2 \rangle \leq \text{Homeo}(X^\omega)$ coincides with Thompsons group F .

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- "Pure" Thomson groups, e.g. F, T, V, V_n, nV_k ,
- Coxeter groups and right-angled Artin groups,
- hyperbolic groups,
- Röyer-Nekrashevich groups for synchronous automata groups,

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Conjecture (Boone-Higman; 1974)

A finitely generated group G has solvable word problem if and only if it embeds in a finitely presented simple group.

Definition (Röver-Nekrashevych groups)

Let $G \leq \text{Aut}(T_X)$ be self-similar. The Röver-Nekrashevych group $V(G)$ is the group of homeomorphisms φ of $\mathcal{C}_X := \partial T_X$ for which there exist finitely many words $v_1, w_1, \dots, v_n, w_n \in X^*$ and element $g_1, \dots, g_n \in G$ such that

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Theorem (Röver; 1999)

If G is Grigorchuk's group, then $V(G)$ is a finitely presented simple group.

Theorem (Belk, Matucci; 2016)

Let G denote Grigorchuk's group. The group $V(G)$ is of type F_∞ .

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