Uniqueness of the minimizer of the normalized volume function

$/\mathbb{C}$ (j/w Chenyang Xu)

$x \in X$ klt singularity (e.g. quotient sing $\mathbb{C}^n/G$, cone over Fano)

i.e. $E \subset Y \xrightarrow{\pi} X$, $\mathrm{ord}_E(K_Y - \pi^* K_X) > -1$

$\mathrm{Val}_{X,x} = \{ \underbrace{\text{valuations}}_{v(\mathbb{C}^*)=0,\ v(fg)=v(f)+v(g),\ v(f+g)\geq \min\{v(f),v(g)\}} \ v : \mathbb{C}(X)^* \to \mathbb{R} \text{ that's } \underbrace{\text{center at } x}_{} \}$ $v(m_x) > 0$

Ex (divisorial val). $E \subset Y \xrightarrow{\pi} X \rightsquigarrow v = c \cdot \mathrm{ord}_E$ $(c>0)$

Log discrepancy $A_X(c\cdot \mathrm{ord}_E) = c \cdot \big(1 + \mathrm{ord}_E(K_Y - \pi^* K_X)\big) > 0$

$\rightsquigarrow$ extend to $\mathrm{Val}_{X,x}$ by lower semi-continuity. $A_X(\cdot) : \mathrm{Val}_{X,x} \to \mathbb{R} \cup \{\infty\}$

Volume $v \in \mathrm{Val}_{X,x} \rightsquigarrow \mathfrak{a}_m(v) = \{ f \in \mathcal{O}_{X,x} \mid v(f) \geq m \}$

$$\mathrm{vol}(v) = \lim_{m\to\infty} \frac{\dim(\mathcal{O}_{X,x}/\mathfrak{a}_m(v))}{m^n/n!} \qquad n = \dim X$$

Rmk $A_X(c\cdot v) = c \cdot A_X(v)$; $\mathrm{vol}(c\cdot v) = c^{-n}\cdot \mathrm{vol}(v)$

Normalized volume (Chi Li) $\widehat{\mathrm{vol}}(v) = \begin{cases} A_X(v)^n \cdot \mathrm{vol}(v), & A_X(v) < \infty \\ \infty, & A_X(v) = \infty \end{cases}$

Prop (Li) (1) $\widehat{\mathrm{vol}}(c\cdot v) = \widehat{\mathrm{vol}}(v)$

(2) $x \in X$ klt $\Rightarrow \inf_{v \in \mathrm{Val}_{X,x}} \widehat{\mathrm{vol}}(v) > 0$

(local volume) **Def** $\mathrm{vol}(x, X) := \inf_{v \in \mathrm{Val}_{X,x}} \widehat{\mathrm{vol}}(v)$

Prop (Liu) $\mathrm{vol}(x,X) = \inf_{\mathfrak{a}} \mathrm{lct}(\mathfrak{a})^n \cdot \mathrm{mult}(\mathfrak{a})$, $\mathfrak{a} = m_x$-primary

Ex $0 \in \mathbb{A}^n$, $v =$ wt blowup with wts $(a_1, \cdots, a_n)$

$A_X(v) = \sum_{i=1}^n a_i$, $\mathrm{vol}(v) = (\prod a_i)^{-1} \Rightarrow \widehat{\mathrm{vol}}(v) = \dfrac{(a_1 + \cdots + a_n)^n}{a_1 \cdots a_n} \geq n^n$

de Fernex–Ein–Mustaţă : $\mathrm{lct}(\mathfrak{a})^n \cdot \mathrm{mult}(\mathfrak{a}) \geq n^n$, $\forall \mathfrak{a} \Rightarrow \mathrm{vol}(0 \in \mathbb{A}^n) = n^n$.

$\widehat{\mathrm{vol}}$ & K-stability $x \in X = C(V) =$ cone over Fano, $x =$ vertex.

ordinary blowup at $x \rightsquigarrow v_0 \in \mathrm{Val}_{X,x}$

Thm (Li, Li–Liu, Li–Xu) $\mathrm{vol}(x,X) = \widehat{\mathrm{vol}}(v_0) \Leftrightarrow V$ is K-ss.

$\Uparrow$

V has Kähler–Einstein metric

Ex $0 \in X = (x_0^2 + \cdots + x_n^2 = 0) \subseteq \mathbb{A}^{n+1}$ = cone over sm quadric

$\Rightarrow \mathrm{vol}(0, X) = 2(n-1)^n < n^n$.

Stable degeneration conjecture (Li, Li-Xu) $x \in X$ klt sing.

Blum → (1) Minimizer exists : $\mathrm{vol}(x, X) = \inf \widehat{\mathrm{vol}}(v) = \widehat{\mathrm{vol}}(v_*)$ for some v_*

(2) Uniqueness : v_* is unique up to rescaling

Xu → (3) Quasi-monomial : v_* is quasi-monomial

(4) Finite generation : $\mathrm{gr}_{v_*} R = \bigoplus_{\lambda \in \Phi} \mathfrak{a}_\lambda(v_*)/\mathfrak{a}_{>\lambda}(v_*)$, $\Phi = v_*(\mathcal{O}_{X,x})$ is f.g.

(5) Stable degeneration : $0 \in \mathrm{Spec}(\mathrm{gr}_{v_*} R)$ is K-ss Fano cone sing.

Assume (4) for some v_*, Li-Xu (3)+(4) ⇒ (2)+(5).

Thm (Xu, –) v_* is unique up to rescaling.

Cor (finite degree formula) $(y, Y) \xrightarrow{\pi} (x, X)$ quasi-étale,

then $\mathrm{vol}(y, Y) = \mathrm{vol}(x, X) \cdot \deg(\pi)$.

Pf Assume π is Galois $\rightsquigarrow G = \mathrm{Gal}(Y/X)$ $(X = Y/G)$

$v_* =$ minimizer for $\mathrm{vol}(y, Y) \overset{\text{Thm}}{\Rightarrow} v_*$ is G-invariant.

$\mathrm{vol}(y, Y) = \mathrm{vol}^G(y, Y) \overset{\text{Liu-Xu}}{=\!=} \mathrm{vol}(x, X) \cdot \deg(\pi)$ #

Cor (effective bound of π_1^{loc}) $|\pi_1^{loc}(X, x)| \leq \dfrac{n^n}{\mathrm{vol}(x, X)}$, $n = \dim X$

Pf Xu, Braun : $\pi_1^{loc}(X, x)$ is finite

$(y, Y) \longrightarrow (x, X)$ universal cover

$\mathrm{vol}(x, X) \cdot |\pi_1^{loc}| = \mathrm{vol}(y, Y) \leq n^n$ Liu-Xu

Cor (boundedness of K-ss Fano) Fix n & $c > 0$. The set of K-ss Fano var. with $\dim = n$ & $\mathrm{vol} = (-K)^n > c$ is bounded.

Rmk proved by Jiang using Birkar's work for BAB.

Pf Liu : K-ss + $(-K)^n > c \Rightarrow \mathrm{vol}(x, X) > c' (>0)$

prev. cor $\Rightarrow |\pi_1^{loc}(x, X)| \leq N = N(n, c)$

Cartier index of $K_X \leq |\pi_1^{loc}(x, X)| \leq N$

boundedness $\Leftarrow$ Hacon - McKernan - Xu #

Pf of thm : main steps (1) Define K-semistable valuation.

(2) K-ss $\Leftrightarrow$ minimizer (3) K-ss val is unique (up to rescaling)

(sloppy) Def An m-basis type divisor of $v \in \mathrm{Val}_{X,x}^{=1}$ $\leftarrow A_X(v) = 1$

$$D = \frac{1}{mN_m} \sum_{i=1}^{N_m} \{f_i = 0\}$$

where $N_m = \dim \mathfrak{a}_m(v)/\mathfrak{a}_{m+1}(v)$

$f_1, \cdots, f_{N_m}$ = basis of $\mathfrak{a}_m(v)/\mathfrak{a}_{m+1}(v)$.

We say v is K-ss if m-basis type div are "asymptotically lc" ($m \to \infty$). (i.e. $\limsup_{m \to \infty} (\inf \mathrm{lct}(D)) \geq 1$)

Prop $X = C(V)$, v_0 is K-ss $\Leftrightarrow$ V is K-ss.

K-ss $\Rightarrow$ minimizers (lct vs vol)

$$\widehat{\mathrm{vol}}(v)^{1/n} = A_X(v) \cdot \mathrm{vol}(v)^{1/n} = \frac{A_X(v)}{\mathrm{vol}(v)^{-1/n}}$$

- If we can find some divisor D s.t. $v(D) = \mathrm{vol}(v)^{-1/n}$ then v_0 = minizer $\Leftrightarrow$ v_0 computes $\mathrm{lct}(X; D)$
- want D to be "of basis type"
- compatible divisor's trick (Ahmadinezhad, $-$)

$\rightsquigarrow$ For any $v_0, v_1 \in \mathrm{Val}_{X,x}$
v_0 minimizer $\Uparrow$ $\widehat{\mathrm{vol}}(v_1) \geq \widehat{\mathrm{vol}}(v_0)$ $\Uparrow$

D "of basis type" (for v_0 & v_1) s.t. $v_0(D) = \mathrm{vol}(v_0)^{-1/n}$, $v_1(D) \geq \mathrm{vol}(v_1)^{-1/n}$.

- if v_0 is K-ss $\approx$ v_0 computes lct of its basis type div.

Minimizer $\Rightarrow$ K-ss. (Li's derivative argument).

v_0 K-ss $\Leftrightarrow$ basis type div are asymp. lc

$$\Leftrightarrow A_X(v) \geq \underbrace{\liminf_{m \to \infty} \left(\sup v(D)\right)}_{= S(v_0; v)} \quad \forall v \in \mathrm{Val}_{X,x}$$

$$\Longleftrightarrow \underbrace{A_X(v) - S(v_0; v)} \geq 0$$

$\hookrightarrow$ realize as derivative of normalized volume function on a larger space