SCMS Seminar

Higher Eisenstein elements, higher Eichler formulas and rank of Hecke algebras

Dr. LECOUTURIER Emmanuel
Paris 7

Lecture 1
Time: 10:00-11:00 am., Monday, Oct.23, 2017
Lecture 2
Time: 10:00-11:00 am., Wednesday, Oct.25, 2017

Venue: Room 2213, East Main Guanghua Tower, Handan Campus

Abstract: In his classical work, Mazur considers the Eisenstein ideal I of the Hecke algebra $\mathbb{T}$ acting on cusp forms of weight 2 and level $\Gamma_0(N)$ where N is prime. When p is an Eisenstein prime, *i.e.* p divides the numerator of $\frac{N-1}{12}$, denote by $\mathbf{T}$ the completion of $\mathbb{T}$ at the maximal ideal generated by I and p. This is a $\mathbf{Z}_p$-algebra of finite rank $g_p \geq 1$ as a $\mathbf{Z}_p$-module.

Shanghai Center for Mathematical Sciences
22F East Guanghua Tower, No.220 Handan Road, Shanghai, China
Tel: 55665643 Fax: 65642190 Postcode: 200433 Email: scms@fudan.edu.cn

SCMS Seminar

Mazur asked what can be said about g_p. Merel was the first to study g_p. Assume for simplicity that $p \geq 5$. Let $\log : (\mathbf{Z}/N\mathbf{Z})^{\times} \to \mathbf{F}_p$ be a surjective morphism. Then Merel proved that $g_p \geq 2$ if and only if $\mathbf{F}_{N^2}^{\times}$

We prove that we have g_p ≥ 3 if and only if

$$\sum_{k=1}^{\frac{N-1}{2}} k \cdot \log(k) \equiv \sum_{k=1}^{\frac{N-1}{2}} k \cdot \log(k)^2 \equiv 0 \ (\text{mod } p).$$

We also give a more complicated criterion to know when $g_p \geq 4$. Moreover, we prove *higher Eichler formulas*. More precisely, let

$$H(X) = \sum_{k=0}^{\frac{N-1}{2}} \binom{\frac{N-1}{2}}{k}^2 \cdot X^k \in \mathbf{F}_N[X]$$

be the classical Hasse polynomial. It is well-known that the roots of H are simple and in $\mathbf{F}_{N^2}^{\times}$. Let L be this set of roots. We prove that

$$\sum_{\lambda \in L} \log(H'(\lambda)) \equiv 4 \cdot \sum_{k=1}^{\frac{N-1}{2}} k \cdot \log(k) \ (\text{modulo } p)$$

and, if $g_p \geq 2$,

$$\sum_{\lambda \in L} \log(H'(\lambda))^2 \equiv 4 \cdot \sum_{k=1}^{\frac{N-1}{2}} k \cdot \log(k)^2 \ (\text{modulo } p).$$

Shanghai Center for Mathematical Sciences
22F East Guanghua Tower, No.220 Handan Road, Shanghai, China
Tel: 55665643 Fax: 65642190 Postcode: 200433 Email: scms@fudan.edu.cn

The proof of these results are based on the theory of higher Eisenstein elements. These are elements of some Hecke module, which have (in particular) the property to be annihilated by some power of the Eisenstein ideal. We consider several Hecke modules in our work and compute explicitly some higher Eisenstein elements in these modules.

Shanghai Center for Mathematical Sciences
22F East Guanghua Tower, No.220 Handan Road, Shanghai, China
Tel: 55665643 Fax: 65642190 Postcode: 200433 Email: scms@fudan.edu.cn