Syzygies of Algebraic Varieties

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These lectures are based on a book that is under preparation together with Rob Lazarsfeld. The first two lectures will be elementary introduction. ~~N~~ In the last two lectures, we'll discuss some recent research.

Lecture I. Introduction and examples.

One of the best ways to input a variety or a module to a computer is to input the presentation of ~~a~~ the module. It was the basic insight of Hilbert that it would be very useful to consider the full resolution of the module.

Theorem of Hilberts if

$G = Gl(r, \mathbb{C})$ (or ~~a~~ more generally a linearly reductive ~~gp~~ group. ~~Let~~ Let V be a representa of G. Set

$S = \mathrm{Symm}(V) \simeq \mathbb{C}[x_1, \ldots x_m]$. Consider

$S^G = R = \{ f \in S \mid g \cdot f = f \}$. Then

S^G is a finitely generate $\mathbb{C}$-algebra.

Hilbert shows that the Hilbert series

$H_R(t) = \sum_0^\infty (\dim R_m) t^m$ is a rational

function.

More generally, suppose M is a finitely generated graded S-module. Set the Hilbert series of M to be

$$H_M(t) = \sum_{m \in \mathbb{Z}} (\dim M_m)\, t^m .$$

~~Let~~ Observe $\dim M_m = 0$ if $m << 0$, since M is finitely generated. Assume $\deg x_i = a_i \in \mathbb{Z}_{>0}$ for $i = 1, \dots, n$.

Theorem. (Hilbert) There is a Laurent polynomial $Q_M(t) \in \mathbb{Z}[t, t^{-1}]$ such that

$$H_M(t) = \frac{Q_M(t)}{\prod_{i=1}^{n}(1 - t^{a_i})} .$$

Proof. First we give an indirect proof by induction n. Consider the exact Complex

$$0 \to \mathrm{Ker} \to M(-d_n) \xrightarrow{x_n} M \to M/x_nM \to 0$$

Observe that Ker and M/x_nM are finitely generated $\mathbb{C}[x_1, \dots, x_{n-1}]$ grade modules.

We have the relation

$$(1 - t^{d_n})\, H_M(t) = H_{M/x_nM}(t) - H_K(t) \quad .$$

By induction, the righ side is of the form $\dfrac{R(t)}{\prod_1^{n-1}(1-t^{d_i})}$. Hence $H_M(t)$ has the desired form.

Suppose now that M has a finite grade free resolution of the form.

$$0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0 \quad \text{---} \ (*)$$

$$H_M(t) = \sum_{i=0}^{n} (-1)^i H_{F_i}(t).$$

$$F_i(t) = \bigoplus_j S(-j)^{\beta_{ij}}$$

$H_{S(-j)}(t)$

$$\cancel{S(-j)} = t^j \cdot H_S(t) = \frac{t^j}{\prod_{i=1}^{n}(1-t^{d_i})}$$

$$H_{F_i}(t) = \sum_j \beta_{ij} \cdot \frac{t^j}{\prod_{i=1}^{n}(1-t^{d_i})}.$$

So from the resolution of M, we can compute $H_M(t)$.

Let assume assume $\deg x_i = a_i = 1$ for $i = 1, \cdots m$.

$H_M(t) = \dfrac{Q_M(t)}{(1-t)^n}$. We can factor

$Q_M(t) = (1-t)^c E_M(t)$, where $E_M(1) \neq 0$.

We can rewrite

$$H_M(t) = \frac{E_M(t)}{(1-t)^d},$$ where $d = n - c$.

Write

$E_M(t) = \sum_{i=r}^{s} b_i t^i$ is called the Euler polynomial of M. Since $\frac{1}{(1-t)^d} = \sum_{j=0}^{\infty} \binom{d-1+j}{d-1} t^j$.

We see for $x \in \mathbb{Z}$ and suffical large.

$$\dim M_x = \sum_{i=r}^{s} b_i \binom{d-1+(x-i)}{d-1} \doteq h_M(x).$$

It is suffical to chose $x \geq s$.

Observe that

$$\dim M_x = h_M(x)$$

$$h_M(x) = \frac{\left(\sum_1^s b_j\right) x^{d-1}}{(d-1)!} + \text{lower terms.}$$

$\sum_1^s b_j = E_M(1) \neq 0$. So $\dim M_x = O(x^{d-1})$.

It follows from dimension theorem, $\dim M = d$. It remains to see that a resolution as in $(*)$ and whether there is a "best" resolution.

Throughout these lectures, we'll assume the base field k is algebraically closed and of characteristic zero.

Let $S = k[x_1, \dots, x_n]$, ~~d~~ when $\deg x_i = 1$. Set $S_+ = \langle x_1, \dots x_n \rangle$ be the unique homogenous ideal in S.

$k = S/S_+$ as a ~~Gr~~ graded S-module.

~~St~~

Lemma 1.2. ~~Let M be~~ ~~a~~ (Graded Nakayama's Lemma) Let M be a finitely generated graded S-Module. The following are equivalent.

(i) $M \neq 0$

(ii) $S_+ M \neq M$

(iii) $M \otimes S/S_+ = M \otimes_S k \neq 0$.

Proof. If $M \neq 0$, then there an integer $l_0 \in \mathbb{Z}$ such that $M_{l_0} \neq 0$, but $M_l = 0$ for $l < l_0$. But $(S_+M)_{l_0} = 0$. So (ii) ⇒ (i).
(ii) ⇒ iii and (iii) ⇒ (i) are clear. $\square$

Corollary 1.3. Let $\varphi : M \to N$ be a morphism of graded S-modules, where N is finitely generated. Then φ is surjective if and only if $\varphi \otimes_S k$ is surjective.

Consider

$$F_1 \xrightarrow{\delta_1} F_0 \xrightarrow{\varepsilon} M \to 0$$

be a presentation by finitely generated graded free S-module. ε corresponds to a minimal set of homogeneous generators if and only $\varepsilon \otimes_S k$ is an isomorphism. Equivalently $\delta_1 \otimes_S k = 0$. This is saying all the entries of matrix of δ_1 are in S_+. By choosing a minimal set of homogeneous generators of $\operatorname{Ker} \varepsilon_1$ etc. We get a "minimal resolution" of the form

$$F_m \xrightarrow{\delta_m} F_{m-1} \to \cdots F_1 \xrightarrow{\delta_1} F_0 \xrightarrow{\varepsilon} M \to 0$$

where $\delta_i \otimes_S k = 0$ for $i > 0$.

Theorem 1.4. (Hilbert's syzygies theorem).
Let M be a finitely generated graded S-module. Then M has a minimal graded free resolution of form

$$0 \to F_n \xrightarrow{\delta_n} \cdots \longrightarrow F_1 \xrightarrow{\delta_1} F_0 \xrightarrow{\varepsilon} M \to 0,$$

where $F_i \otimes k = \mathrm{Tor}_i(M, k)$ as a finite dimensional graded vector space. Furthermore, any two minimal graded free resolutions of M are isomorphic.

Proof. Let

$$\cdots F_{n+1} \xrightarrow{\delta_n} F_n \to \cdots \xrightarrow{\delta_1} F_0 \xrightarrow{\varepsilon} M$$

be a minimal graded free resolution of M. Then $F_i \otimes k \cong \mathrm{Tor}_i(M, k)$, since $\delta_i \otimes_S k = 0$. To show $F_{n+1} = 0$, we just need to show $\mathrm{Tor}_{n+1}(M, k) = 0$. One can also a resolution of k by the Koszul complex. Set $V = S_1 = \bigoplus_i k x_i$.

$$K_\cdot \; 0 \to \Lambda^n V \otimes S(-n) \to \cdots \cdots \to V \otimes S(-1) \to S \to k \to 0$$

is a resolution of k. $\mathrm{Tor}(M, k)$ is computed by $K_\cdot \otimes M$. Hence $\mathrm{Tor}_{n+1}(M, k) = 0$ as desired.

Graded Betti numbers of M

Suppose that ith piece of the resolution $F_i = \bigoplus S(-j)^{\beta_{ij}}$.

Example 1.15 for the twistal cubic curve $C \subseteq \mathbb{P}^3$. The homogenous ideal

$S = k[x_1, x_2, x_3, x_4]$

$$I_C = \langle \underbrace{x_1x_3 - x_2^2}_{Q_1}, \underbrace{-x_1x_4 + x_2x_3}_{Q_2}, \underbrace{x_2x_4 - x_3^2}_{Q_3} \rangle.$$

which the generators are given by the (2×2)-minors of $\begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{pmatrix}$.

The resolution of the homogenous ideal I_C

The resolution of I_C

$$\begin{array}{ccccccc} & F_1 & & F_0 & & & \\ & \| & & \| & & & \\ 0 \to & 2S(-3) & \xrightarrow{\delta_1} & 3S(-2) & \xrightarrow{\varepsilon} & I_C & \to 0 \end{array}$$

$$\delta_1 = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \\ x_3 & x_4 \end{pmatrix}, \qquad \varepsilon = (Q_1, Q_2, Q_3)$$

Observe that

$$\det \begin{pmatrix} x_1 & x_1 & x_2 \\ x_2 & x_2 & x_3 \\ x_3 & x_3 & x_4 \end{pmatrix} = 0 \qquad \det \begin{pmatrix} x_2 & x_1 & x_2 \\ x_3 & x_2 & x_3 \\ x_4 & x_3 & x_4 \end{pmatrix}$$

Expand using the first columns, one see $\delta_1 \circ \varepsilon = 0$.

. In this case $\beta_{0,2} = 3$ and $\beta_{1,3} = 2$.

Otherwise, they are zero.

Exactnes follows from the the Hilbert Burch theorem about resolution of the homogens ideal I_X of ~~codi~~ a codimension two variety when the projective dimension of I_X is one. Then I_X is given by the $(n-1)\times(n-1)$ minors of of a $n\times n-1$ matrices of homogen polynomials. For details, see Eisenbud's book.

In the example we are resolving I_X where X is the cone of twisted cubic cubic in $\mathbb{A}^4$.

Example $\cancel{5}$ 1.6. Take $S = k[x,y]$.

Let $I = \langle xy, x^3, y^5 \rangle$.

Then S/I has a resolution

$$0 \to S(-4) \oplus S(-6) \xrightarrow{B} S(-2) \oplus S(-3) \oplus S(-5) \to S \to S/I \to$$

when $B = \begin{pmatrix} x^2 & y^4 \\ -y & 0 \\ 0 & -x \end{pmatrix}$. One sees the 2x2 minors generated the ideal I.

This is ~~an~~ another illustration of the Hilbert - Burch theorem in the homogen setting.

$\beta_{00}(S/I) = 1$, $\beta_{1,2} = \beta_{13} = \beta_{15} = 1$

$\beta_{2,4} = 1$ $\beta_{2,6} = 1$.

Example 1.7 Rank 1 locus of a generic $2 \times n$ matrix.

Let A be a n-dimensional vector space and ~~a~~ B be a 2-dimensional vector space. We would like to understand the syzygies of the rank $\leq$ 1 locus of $\mathrm{Hom}(A, B) \simeq A^* \otimes B$.

We consider $\mathbb{P}(\mathrm{Hom}(A, B)) = \mathbb{P}(\mathrm{Hom}(A,B)^*) = \mathbb{P}((B^* \otimes A) \simeq \mathbb{P}^{2n-1}$. First $\mathbb{P}$ is the projective space of lines. The second $\mathbb{P}$ we think of as space of 1-dimensional quotient.

On $\mathbb{P}^{2n-1}$, we have a surjection

$$\mathcal{O}_{\mathbb{P}}\, B^* \otimes A \to \otimes\, \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_{\mathbb{P}}(1) \to 0$$

This induces the universal map

$$A \otimes \mathcal{O}_{\mathbb{P}^{2n-1}} \xrightarrow{\ \varphi\ } B \otimes \mathcal{O}_{\mathbb{P}^{2n-1}}(1)$$

$\varphi = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ x_{21} & \cdots & x_{2n} \end{pmatrix}$ x_{ij}'s are the coordinates of $\mathbb{P}^{2n-1}$. Set

$$\mathbb{P} = \mathbb{P}(B \otimes \mathcal{O}_{\mathbb{P}^{2n-1}}) \simeq \mathbb{P}^{2n-1} \times \mathbb{P}(B) = \mathbb{P}^{2n-1} \times \mathbb{P}^1$$

Let π_1 and π_2 be the two projection maps. Denote by $\mathcal{O}_{\mathbb{P}}(a,b)$ the line bundle $\pi_1^*\mathcal{O}_{\mathbb{P}^{2n-1}}(a) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^1}(b)$. On $\mathbb{P}$, we have the following diagram

$$\begin{tikzcd}
\pi^*(A \otimes \mathcal{O}_{\mathbb{P}^{2n-1}}) \arrow[r, "\varphi"] \arrow[dr, "\beta"'] & \pi^* B \otimes \mathcal{O}_{\mathbb{P}^{2n-1}}(1) \arrow[d, "\psi"] \\
 & \pi^* B \otimes \mathcal{O}_{\mathbb{P}}(1,1) \arrow[d] \\
 & 0
\end{tikzcd}$$

Let Y be the zero set of β

For each point $q \in \mathbb{P}(B)$. We are asking to find the $M \in Hom(A, B), -\{0\}$ such that the composition map

$$A \xrightarrow{M} B \to q \to 0$$

is the zero map. Let $S \subseteq B$ be the kernel of $B \to q \to 0$. Then the solution set is just $\mathbb{P}(\cancel{A}, Hom(A, S))$. We see Y is a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}(B)$.

We see $\dim Y = n$. $\dim \mathbb{P}^{2n-1} \times \mathbb{P}^1 = 2n$.

So Y has codimension n in $\mathbb{P}^{2n-1} \times \mathbb{P}^1$.

So locally $A \otimes \mathcal{O}_{\mathbb{P}}(-1,-1) \to \mathcal{O}_{\mathbb{P}}$ give a regular sequence of length n.

On $\mathbb{P}^{2n-1} \times \mathbb{P}^1$, we have a

Koszul complex

$$0 \to \Lambda^n A \otimes \mathcal{O}(-n,-n) \to \cdots \Lambda^2 A \otimes \mathcal{O}(-2,-2) \xrightarrow{\beta_2} A \otimes \mathcal{O}(-1,-1) \xrightarrow{\beta_1} \mathcal{O}_{\mathbb{P}} \xrightarrow{\beta} \mathcal{O}_Y$$

Set $X = \pi_1(Y)$. is the precise

the rank one locus in $\mathbb{P}(\mathrm{Hom}(A,B))$.

On ~~each $x \in X$~~ Consider, ~~the~~ $p: Y \to X$.

$x \in X$, the fiber is just $\mathbb{P}(\mathrm{coker}\,\varphi_x) = \mathbb{P}^0$

So $X \simeq Y$. Further one checks

$\pi_{2*}\mathcal{O}(1,0) \cong \pi_{2*}\mathcal{O}_X(1,0) \simeq A \otimes \mathcal{O}_{\mathbb{P}^1}$.

We see $Y \simeq \mathbb{P}(A) \times \mathbb{P}(B)$.

In term of linear algebra, it says rank one map M from A to B is determine ~~Ker M~~ Im M a ~~n~~ ~~dimens subspace of~~ 1-dimens quotient space of A, and regard Im M as a 1-dimens subsp of B. Here X is isomorph to $\mathbb{P}(A) \times \mathbb{P}(B)$.

Apply π_{1*} to Koszul complex

we obtain

$$0 \to \pi_{1*}\operatorname{Ker}\beta_0 \to \pi_{1*}\mathcal{O}_{\mathbb{P}} \xrightarrow{\beta_0} \pi_{1*}\mathcal{O}_{\ell} = \mathcal{O}_{\Delta} \to 0$$
$$\pi_{1*}\operatorname{Ker}\beta_0 = \mathcal{I}_X, \qquad \pi_{1*}\mathcal{O}_{\mathbb{P}} = \mathcal{O}_{\mathbb{P}^{2n-1}}$$

From $0 \to \operatorname{Ker}\beta_1 \to A(\mathcal{O}_{\mathbb{P}}(-1,-1)) \to \operatorname{Ker}\beta_0 \to$

We see ~~π_1 Ker~~ $\pi_{1*}(\operatorname{Ker}\beta_0) = R^1\pi_* \operatorname{Ker}\beta_1$.

Chasing further

$$0 \to R^1\pi_{1*}\Lambda^n A\,\mathcal{O}(-n,-n) \to \cdots \to R^1\pi_{1*}(\Lambda^2 A \otimes \mathcal{O}(-2,-2)) \to R^1\pi_{1*}\operatorname{Ker}\beta_1 \to 0 \qquad \text{— (1.7.1)}$$

is exact.

By Künneth formula, we see

$$R^1\pi_{1*}\mathcal{O}_{\mathbb{P}}(-j,j) = H^1(\mathcal{O}_{\mathbb{P}(B)}(-j)) \otimes \mathcal{O}_{\mathbb{P}^{2n-1}}(-j).$$
$$\simeq (S^{j-2}B)^* \otimes (\det B)^* \simeq (S^{n-2}B)^* \otimes (\Lambda^2 B)^*.$$

hyperplane class from $\mathbb{P}^{n-1}$ and H_2 be the pull back of the hyperplane class of $\mathbb{P}^1$.

Then $\deg X = (H_1 + H_2)^n = \binom{n}{n-1} H_1^{n-1} H_2 = n.$

since $H_2^j = 0$ for $j \geq 2$ and $H_1^{\ell} = 0$ for $\ell \geq n$.

So ~~X~~ $\deg X = \overset{\overset{n-1}{\|}}{\operatorname{codim}_{\mathbb{P}^{2n-1}}(X)} + 1.$

This is an example of a variety of minimal degree.