I. Representations of $\mathbb{G}_m$

1-dim'l rep of $\mathbb{G}_m$ = homom of alg gps $\mathbb{G}_m \to \mathbb{G}_m$

So: of the form $z \mapsto z^n$ for some $n \in \mathbb{Z}$

Goal: understand higher dim'l representations

Recall from linear algebra
V fin dim'l vec sp, $T: V \to V$ linear endo.

$\forall \lambda \in \Bbbk$, the <u>$\lambda$-eigenspace</u> of T is $V_\lambda = \{v \in V \mid (T - \lambda\, id)v = 0\}$

the <u>generalized λ-eigenspace</u> is
$$V_\lambda^{gen} = \{v \in V \mid (T - \lambda\, id)^n v = 0 \text{ for some } n > 0\}$$

<u>Thm (Jordan decomposition)</u>

$$(*) \quad V = \bigoplus_{\lambda \in \Bbbk} V_\lambda^{gen}$$

<u>Lemma</u> If $S, T: V \to V$ commute, then each one preserves the other's generalized eigensp.

Consider the decomp $(*)$ with respect to T. S preserves each V_λ^{gen} so
$$V_\lambda^{gen} = \bigoplus_{\mu \in \Bbbk} V_{\lambda,\mu}^{gen}$$
where $V_{\lambda,\mu}^{gen} = \{v \in V_\lambda^{gen} \mid (S - \mu\, id)^n v = 0 \text{ for some } n > 0\}$

Combine as λ varies:
$$V = \bigoplus_{\lambda,\mu \in \Bbbk} V_{\lambda,\mu}^{gen}$$

- This generalizes to any (possibly infinite) family of commuting linear operators $V \to V$

Suppose we have a rep of $\mathbb{G}_m$ on V.
$\pi: \mathbb{G}_m \to GL(V)$.

$\mathbb{G}_m$ commutative $\Rightarrow$ $\pi(z), \pi(y)$ commute $\forall z, y \in \mathbb{G}_m$.

Take the shared generalized eigensp. decomposition of V for the family $\{\pi(z) \mid z \in \mathbb{G}_m\}$

We obtain
$$V = \bigoplus_\chi V_\chi^{gen}$$
where $\chi : \mathbb{G}_m \to \Bbbk$ is any function
and $V_\chi^{gen} = \{v \in V \mid (\pi(z) - \chi(z)\,id)^n v = 0$ for some $n > 0$, $\forall z \in \mathbb{G}_m\}$.

Can also define $V_\chi \subset V_\chi^{gen}$ by
$$V_\chi = \{v \in V \mid (\pi(z) - \chi(z)\,id)v = 0 \ \ \forall z\}$$

What can χ be?
- $\pi(z)$ is invertible so $\chi(z) \neq 0 \ \forall z$.
- V_χ is an algebraic subrep, so

$\chi : \mathbb{G}_m \to \Bbbk^\times = \mathbb{G}_m$ is a homom of alg gps

So $\chi(z) = z^m$ for some $m \in \mathbb{Z}$.

Prop. Given a $\mathbb{G}_m$-rep V and a homom $\chi : \mathbb{G}_m \to \mathbb{G}_m$, we have
$$V_\chi = V_\chi^{gen}$$

In other words: all generalized eigensp of $\pi(z)$ are true eigensp.

OR: All the $\pi(z)$ are diagonalizable linear operators

Proof. Without loss of generality assume $V = V_\chi^{gen}$.

Induction on $\dim V$:

$\dim V = 1$ generalized eigensp is automatically a true eigensp.

$\dim V > 1$ Assume $V_\chi \neq V_\chi^{gen}$.

Look for contradiction.

Let $V' = V_\chi \subset V$. $V'' = V/V'$.

$\dim V'' < \dim V$.

In the $\mathbb{G}_m$-rep on V'', each $\pi(z)$ acts with generalized eigenvalue $\chi(z)$, so by induction,
$$V''_\chi = (V'')_\chi^{gen} = V''.$$

Let $v \in V''$ be a nonzero vector, choose $\tilde{v} \in V$ that maps to v.
Since $\pi(z)v = \chi(z)v$ in V'',
$$\pi(z)\tilde{v} = \chi(z)\tilde{v} + (\text{something in } V').$$

Choose a basis $v_1, \cdots, v_k$ for V'.
$$\pi(z)\tilde{v} = \chi(z)\tilde{v} + c_1(z)v_1 + c_2(z)v_2 + \cdots + c_k(z)v_k$$

$$\pi(z)\tilde{v} = \chi(z)\tilde{v} + c_1(z)v_1 + \cdots + c_k(z)v_k$$

$\forall z \in \mathbb{G}_m$

algebraic fns $\mathbb{G}_m \to k$
i.e. Laurent polynomials in z

$$\pi(z^2)\tilde{v} = \chi(z)^2\tilde{v} + c_1(z^2)v_1 + \cdots + c_k(z^2)v_k.$$

But also: $\pi(z^2)\tilde{v} = \pi(z)(\pi(z)\tilde{v})$

$$= \chi(z)^2 + 2\chi(z)c_1(z)v_1 + \cdots + 2\chi(z)c_k(z)v_k$$

$$\Rightarrow \quad c_i(z^2) = 2\underset{z^m}{\chi(z)}\, c_i(z)$$

Exercise This implies that $c_i(z) = 0$

So $\quad \pi(z)\tilde{v} = \chi(z)\tilde{v}$

So $\quad \tilde{v} \in V_\chi = V' \subset V$ - contradiction $\square$.

Thm. Given a $\mathbb{G}_m$-rep V, we have

$$V = \bigoplus_{\chi : \mathbb{G}_m \to \mathbb{G}_m} V_\chi$$

II. Tori

Defn. A (algebraic) torus is an alg gp isomorphic to $\mathbb{G}_m \times \cdots \times \mathbb{G}_m$.

A character of a torus T is a homom. of alg gps $T \to \mathbb{G}_m$.

$X(T)$ = set of all characters.

called the character group or character lattice

If $\chi, \psi \in X(T)$, define a new character $\chi + \psi$ by

$$(\chi + \psi)(\underset{T}{t}) = \chi(t)\psi(t).$$

If $T = \mathbb{G}_m \times \cdots \times \mathbb{G}_m$, then every character is of the form

$$\underset{\mathbb{G}_m \times \cdots \times \mathbb{G}_m}{(t_1, \cdots, t_n)} \longmapsto t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n}$$

for some $(a_1, \cdots, a_n) \in \mathbb{Z}^n$

So

$$X(T) = \mathbb{Z}^n = \text{a free abelian group}$$

depends on the isom $T \simeq \mathbb{G}_m \times \cdots \times \mathbb{G}_m$.

Fact. Any connected subgps & any quotient of a torus is a torus.

Example. Let $T = \left\{ \begin{bmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{bmatrix} \right\} \subset GL_n$

$\simeq \mathbb{G}_m \times \cdots \times \mathbb{G}_m.$

& $X(T) = \mathbb{Z}^n.$

Let $T' = T \cap SL_n$

$= \left\{ \begin{bmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{bmatrix} \middle| \ t_1 \cdots t_n = 1 \right\}.$

T' is again a torus, &

$X(T') = \mathbb{Z}^n / \mathbb{Z}\cdot(1, \cdots, 1)$ - a free abelian group

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Thm. Let T be a torus & let V be a T-repn. Then

$$V = \bigoplus_{\chi \in X(T)} V_\chi$$

where $V_\chi = \{ v \in V \mid (\pi(t) - \chi(t)\,\mathrm{id})\, v = 0 \}.$

Terminology V_χ is called the χ-weight space of V.

If $V_\chi \neq 0$, χ is called a weight of V.

(Could define V_χ^{gen}. Content of thm: $V_\chi = V_\chi^{gen}$.)

III. Formal characters

$\mathbb{Z}[X(T)]$ - group ring of $X(T)$.

For $\lambda \in X(T)$, let e^λ = corresponding element of $\mathbb{Z}[X(T)]$.

"e" doesn't mean anything, just a symbol.

$$e^{\lambda+\mu} = e^\lambda e^\mu \quad \text{in } \mathbb{Z}[X(T)].$$

Elements of $\mathbb{Z}[X(T)]$ look like finite sums $\displaystyle\sum_{\lambda \in X(T)} a_\lambda e^\lambda$, $a_\lambda \in \mathbb{Z}$

Defn Given a T-rep. V, its formal character is

$$\mathrm{ch}\, V = \sum_{\chi \in X(T)} (\dim V_\chi) e^\chi \in \mathbb{Z}[X(T)]$$

Thm. For any two T-repns V_1, V_2, we have

$V_1 \simeq V_2$ as T-reps

$\Leftrightarrow$ $\mathrm{ch}\, V_1 = \mathrm{ch}\, V_2$ in $\mathbb{Z}[X(T)]$.

IV. Application to SL_2

Let $T = \left\{ \begin{bmatrix} a & \\ & a^{-1} \end{bmatrix} \right\} \subset SL_2$

$\cong \mathbb{G}_m \qquad X(T) \simeq \mathbb{Z}$.

For any SL_2-rep V, can regard it as a T-rep, & compute $\mathrm{ch}\, V$.

<u>Example</u>. $V_n = \mathrm{span}\{x^n, x^{n-1}y, \dots, y^n\}$.

$$\begin{bmatrix} a & \\ & a^{-1} \end{bmatrix} \cdot x^{n-i} y^i = a^{n-2i} x^{n-i} y^i$$

<u>weight vector</u> of weight $n-2i$.

So

$$\mathrm{ch}\, V_n = e^n + e^{n-2} + e^{n-4} + \cdots + e^{-n}$$

Recall: L_n = smallest subrep of V_n that contains x^n.

$$\mathrm{ch}\, L_n = e^n + \left(\text{other terms with weight } < n\right).$$

So:

$L_n \not\cong L_m$ if $n \neq m$, because $\mathrm{ch}\, L_n \neq \mathrm{ch}\, L_m$

Yesterday: showed:

- every irreducible SL_2-rep is isom. to some L_n.

Still need to show: every L_n is irreducible.

Know: irreducible SL_2-reps are determined by their highest wts.

If L_n not irreducible, it must have some irreducible subquotient with highest weight n. But that irred must be isom to L_n.

<u>Thm</u>. The irreducible reps of SL_2 are precisely the L_n. $\square$