2021/1/7

Fudan

Inversion of adjunction for quotient singularities.

/ $k = \bar{k}$, ch = 0.

j.w.w. K. Shibata

$$\left(\text{?} \quad \underbrace{mld_x}_{\substack{\uparrow \\ \text{invariant of sing} \\ \text{(in context of MMP)}}}(X, H) = mld_x H \quad \text{for} \quad \begin{array}{l} X : \text{quot sing} \\ \cup \\ H : \text{Cart div} \\ \Downarrow \\ x \end{array} \right)$$

<u>MMP</u> : Termination of flips ← still open

<u>Shokurov</u> introduced MLD.
proposed 2 conjs.

- $\underbrace{\text{ACC conj (Ascending chain cond)} + \text{LSC conj (lower semi-conti)}}_{\text{local prob}} \overset{\text{Shokurov}}{\Longrightarrow} \underbrace{\text{Termination of flips}}_{\text{global prob}}$

- PIA conj : precise version of inversion of adjunction

  ⇝ useful for induction argument.
  (on dim)

Def

- (X, Δ) : log pair i.e. X : normal var
  ($\Delta \geq 0$: $\mathbb{R}$-div s.t. $K_X + \Delta$: $\mathbb{R}$-Cart)

  ideal ver or
  $(X, \mathfrak{a})$: log pair i.e. X : $\mathbb{Q}$-Gor normal var
  $\mathfrak{a} = \prod_{i=1}^{s} \mathfrak{a}_i^{r_i}$: $\mathbb{R}$-ideal sheaf
  ($r_i \geq 0$, $0 \neq \mathfrak{a}_i \subset \mathcal{O}_X$ coh. ideal)

- For E : div over X i.e. $X' \xrightarrow{f} X$: proper birat mor
  ($E \subset X'$: prime div on X')

  $\rightsquigarrow$ • $a_E(X, \Delta) := 1 - \mathrm{coeff}_E \Delta'$, where we define Δ' by: $K_{X'} + \Delta' = f^*(K_X + \Delta)$
  (log discrepancy)

  ideal ver
  • $a_E(X, \mathfrak{a}) := 1 + \mathrm{coeff}_E(K_{X'} - f^*K_X) - \underbrace{\mathrm{ord}_E \mathfrak{a}}_{\sum_i r_i \, \mathrm{ord}_E \mathfrak{a}_i}$

- $x \in X$ cl.pt $\rightsquigarrow$ $\mathrm{mld}_x(X, \Delta) := \inf\limits_{\substack{E : \text{div over } X \\ c_X(E) = \{x\}}} a_E(X, \Delta)$ (Fact $\in \mathbb{R}_{\geq 0} \cup \{-\infty\}$
  if mld ≥ 0, "inf = min".)

  $f(E) = c_X(E)$

there is no infinite decreasing seq.

<u>ACC conj</u> $d > 0$, $I \subset [0,1]$: DCC set : Fixed

$\left(\Delta = \sum a_i D_i \in I \overset{\text{def}}{\iff} a_i \in I \ \forall i \right)$

$\rightsquigarrow A(d, I) := \{ \mathrm{mld}_x (X, \Delta) \mid \dim X = d, \Delta \in I, x \in X \}$ satisfies ACC.

<u>LSC conj</u> (X, Δ) : log pair $\rightsquigarrow$

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{R}_{\geq 0} \cup \{-\infty\} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ x & \longmapsto & \mathrm{mld}_x (X, \Delta) \end{array}$$

is lower semi-conti.

<u>PIA conj</u> (X, Δ) : log pair

$\cup$
S : normal Cart div
$\rotatebox{90}{$\in$}$ s.t. $S \not\subset \mathrm{Supp}\,\Delta$
x

$\rightsquigarrow \mathrm{mld}_x (X, \Delta + S) = \mathrm{mld}_x (S, \Delta|_S)$.

cf. <u>BDD conj</u> : $\forall d > 0 \ \exists a(d) \in \mathbb{R}$ s.t. $\forall_{x \in} X$: d-dim $\mathbb{Q}$-Gor $\mathrm{mld}_x X \leq a(d)$.

(Known only for $d \leq 3$)

(∘ ACC conj for $I = \emptyset$ $\Rightarrow$ BDD conj.
∘ LSC conj $\Rightarrow$ BDD conj $a(d) = d$.)

cf. <u>Thm</u> (Shokurov) ACC conj + LSC conj $\Rightarrow$ termination of flips.

Known results

ACC : ① $d \le 2$: [Alexeev], [Shokurov] ← by classification of surf sing

② X : fixed (klt) var, I : finite : [Kawakita] ← application of ACC for LCT (rationality of accum pt)

③ X : 3-dim sm, I : DCC $\rightsquigarrow$ ACC on $[0,1]$: [Kawakita]

④ X : 3-dim canonical, I : finite : [N]

⑤ X : 3-dim, $I = \{0\}$ $\rightsquigarrow$ ACC around 1 : [Jiang] (1-gap theorem)

LSC : ① $d \le 3$: [Ambro]

② X : smooth : [Ein, Mustaţă, Yasuda] } ← jet schm theory

③ X : LCI : [Ein, Mustaţă]

④ X : quot sing : [N] ← Yasuda's twisted jet stack

$\boxed{2} \Leftarrow \boxed{1}$ + LSC④

$x \mapsto \underbrace{\mathrm{mld}_x(S, \mathfrak{a})}$: LSC?

$\overset{\boxed{1}}{=}$ $\mathrm{mld}_x(X, \mathfrak{b}\mathcal{O}_x(-S))$: LSC by ④

PIA : ① $d \le 3$:

② X : smooth : [EMY] } jet schm theory

③ X : LCI : [EM]

Today : $\boxed{1}$ PIA for quot sing : $X : \mathbb{A}^N/G$ ← · free in codim 1 · $G \subset GL_n(k)$, $S \subset X$: klt Cartier prime div, $\mathfrak{a}$: $\mathbb{R}$-ideal on X.

$\boxed{2}$ LSC for hyper quot sing : $(S, \mathfrak{a})$

$\boxed{3}$ ACC for quot sing (without boundary) : $A_{quot}(d) := \left\{ \mathrm{mld}_x(X) \;\middle|\; \begin{array}{l} X : \text{quot sing}, \\ \ \ \rotatebox{90}{$\in$} \\ x \qquad \dim X = d \end{array} \right\}$: ACC.

Idea of proof of $\boxed{\mathrm{I}}$: combining [EMY 03] + [DL 02]

① Ein - Mustaţă - Yasuda's proof for PIA for $\underbrace{H}_{\text{div}} \subset \underbrace{X}_{\text{sm}}$

- describe $\mathrm{mld}_x(H.\mathfrak{a})$, $\mathrm{mld}_x(X.\mathfrak{b})$ in terms of arc spaces. (codim of cylinders $\subset H_\infty . X_\infty$)
- compare their descriptions.

$\downarrow$ this method does not work when X : sing.

② Denef - Loeser's theory of the arc spaces of quotient vars

- $(Y/G)_\infty$ can be studied by W_∞ for some $k[t]$-var W.
- If $Y = \mathbb{A}^N \rightsquigarrow W_\infty = (\mathbb{A}^N)_\infty$

$\rightsquigarrow$ $\mathrm{mld}_x(X.\mathfrak{b})$ can be described in terms of $\underbrace{\mathbb{A}^N_\infty}_{\text{sm}}$.

$\Uparrow$

[EMY]'s method works!

$$\begin{array}{ccc} H & \subset & X \\ \parallel & & \parallel \\ S/G & \subset & \mathbb{A}^N/G \end{array}$$

jet schm · arc space

X : schm / $\Bbbk$

$\rightsquigarrow$ • X_m : the m-th jet schm $\overset{\text{set}}{=} \{ \operatorname{spec} \Bbbk[t]/(t^{m+1}) \to X \}$

• X_∞ : the arc space $\overset{\text{set}}{=} \{ \operatorname{Spec} \Bbbk[[t]] \to X \}$

• $(m \geq n)$ $\pi_{m,n} : X_m \to X_n$: truncation map. (induced by $\Bbbk[t]/t^{m+1} \to \Bbbk[t]/t^{n+1}$)

• $X_\infty \underset{\text{def}}{=} \varprojlim_m X_m$, $\varphi_m : X_\infty \to X_m$

Cylinder : $\underset{\text{cylinder}}{C \subset X_\infty}$ $\underset{\text{def}}{\iff}$ $\exists m \geq 0$ $\exists \underset{\text{constructible subset}}{S \subset X_m}$ s.t. $C = \varphi_m^{-1}(S)$.

$\downarrow$ typical example

Contact loci : $\mathfrak{a} \subset \mathcal{O}_X$: ideal

Def
$\operatorname{ord}_\gamma \mathfrak{a} := \sup \{ r \geq 0 \mid \gamma^*(\mathfrak{a}) \subset (t^r) \}$

$\rightsquigarrow$ • $\operatorname{Cont}^m(\mathfrak{a}) := \{ \gamma \in X_\infty \mid \operatorname{ord}_\gamma \mathfrak{a} = m \}$

• $\operatorname{Cont}^{\geq m}(\mathfrak{a}) := \{ \quad \mid \quad \geq m \}$ $\left(= \varphi_{m-1}^{-1}(Z_{m-1}) \right)$ if $Z = V(\mathfrak{a}) \subset X$ $\left(\underset{\text{closed}}{Z_{m-1} \subset X_{m-1}} \right)$

Codim of cylinder : X : var / $\Bbbk$, $C \subset X_\infty$: cylinder, $\operatorname{Jac}_X := \operatorname{Fitt}^n \Omega_X$ $(n := \dim X)$

• when $C \subset \operatorname{Cont}^e(\operatorname{Jac}_X)$: $\operatorname{codim} C \overset{(m \gg 0)}{:=} (m+1)\dim X - \dim \varphi_m(C)$

• general case : $\operatorname{codim} C := \min_{e \geq 0} \operatorname{codim}(C \cap \operatorname{Cont}^e \operatorname{Jac}_X)$

well-def.

<u>Fact</u> : $\overset{(m \geq e)}{\varphi_{m+1}(\operatorname{Cont}^e(\operatorname{Jac}_X))} \to \varphi_m(\operatorname{Cont}^e(\operatorname{Jac}_X))$ is piecewise trivial fibr w/ fiber $\cong \mathbb{A}^n$.

Ein–Mustaţă–Yasuda's proof of PIA for sm var

Def ($\mathfrak{n}_{r,X}$) $\quad \mathrm{Im}\,\varphi = \mathfrak{n}_{r,X}\mathcal{O}_X(rK_X)$

$(\Omega_X^n)^{\otimes r} \xrightarrow{\varphi} \underbrace{\mathcal{O}_X(rK_X)}_{\text{invertible}}$

Thm [EMY] [EM] $(X, \mathfrak{a}^\delta)$: log pair, rK_X : Cartier $(r>0)$,

$$\rightsquigarrow \mathrm{mld}_x(X, \mathfrak{a}^\delta) = \inf_{w,\ell \geq 0}\left\{ \mathrm{codim}\left(\mathrm{Cont}^{w}(\mathfrak{a}) \cap \mathrm{Cont}^{\ell}(\mathfrak{n}_{r,X}) \cap \mathrm{Cont}^{\geq 1}(\mathfrak{m}_x)\right) - \frac{\ell}{r} - \delta w \right\}$$

$$\left(= \inf_{w,\ell \geq 0}\left\{ \mathrm{codim}\left(\mathrm{Cont}^{\geq w}(\mathfrak{a}) \cap \mathrm{Cont}^{\ell}(\mathfrak{n}_{r,X}) \cap \mathrm{Cont}^{\geq 1}(\mathfrak{m}_x)\right) - \frac{\ell}{r} - \delta w \right\} \right)$$

NOTE: $\mathfrak{n}_{1,X} = \mathcal{O}_X$ if X : sm

$(\mathfrak{n}_{1,X} = \mathrm{Jac}_X$ if X : l.c.i $)$

cf.

Thm (Denef–Loeser 99)

$\underset{\text{sm}}{X'} \xrightarrow{f} X$: proper birat mor

Def (Jac_f) $\quad \mathrm{Im}\,\varphi = \mathrm{Jac}_f \otimes \Omega_{X'}^n$

$f^*\Omega_X^n \xrightarrow{\varphi} \underbrace{\Omega_{X'}^n}_{\text{inv}}$

$$\begin{array}{ccc} X'_\infty & \xrightarrow{f_\infty} & X_\infty \\ \cup & & \cup \\ \underset{\text{cylinder}}{B} & & A := f_\infty(B) \end{array}$$

Suppose $\quad B \subset \mathrm{Cont}^{e''}\mathrm{Jac}_{X'} \cap \mathrm{Cont}^{e}\mathrm{Jac}_f$

$A \subset \mathrm{Cont}^{e'}\mathrm{Jac}_X$

for some $e, e', e'' \geq 0$

$\rightsquigarrow \mathrm{codim}\, B + e = \mathrm{codim}\, A$

$$\begin{array}{ccc} f^*\Omega_X^n & \xrightarrow[\mathrm{Jac}_f]{} & \Omega_{X'}^n \\ \downarrow \otimes \mathfrak{n}_X & & \downarrow \wr \\ f^*\mathcal{O}(K_X) & \xrightarrow[\text{discrepancy}]{} & \mathcal{O}(K_{X'}) \end{array}$$

want to show : $\mathrm{mld}_x(X, \mathfrak{a}^{\delta}\mathcal{O}_X(-H)) \overset{?}{\geq} \mathrm{mld}_x(H, \mathfrak{a}|_H^{\delta})$ ($\leq$: always true)

setting ($x \in \underset{\text{hyper surf}}{H} \subset \underset{\text{sm}}{X}$)

$(\exists v, w)$ $\|$ Thm $\qquad\qquad$ $\wedge|$ Thm

$$\mathrm{codim}(\underset{\shortparallel}{C_{v,w}}) - \delta w - v \xleftarrow{\text{want to}} \xrightarrow{\text{compare}} \mathrm{codim}(\underset{\shortparallel}{D_{w,l}}) - l - \delta w$$

$$\left(\mathrm{Cont}^{\geq w}(\mathfrak{a}) \cap \mathrm{Cont}^{\geq v}(H) \cap \mathrm{Cont}^{\geq 1}(\mathfrak{m}_x)\right) \quad \left(\mathrm{Cont}^{\geq w}(\mathfrak{a}|_H) \cap \mathrm{Cont}^{l}(\mathrm{Jac}_H) \cap \mathrm{Cont}^{\geq 1}\mathfrak{m}_x\right)$$

$\subset X_\infty$? $\qquad\qquad$ $\subset H_\infty$

Key claim

① If $C_{v,w} \cap H_\infty \cap \mathrm{Cont}^{l}(\mathrm{Jac}_H) \neq \emptyset \Rightarrow \mathrm{codim}(C_{v,w} \cap H_\infty) \leq \mathrm{codim}\, C_{v,w} + l - v$

② $\exists\, l$ s.t. $C_{v,w} \cap H_\infty \cap \mathrm{Cont}^{l}(\mathrm{Jac}_H) \neq \emptyset$. $\left(\underset{\text{Fact}}{\Longleftrightarrow} \quad \underset{\text{sm}}{W} \xrightarrow{g} H : \forall \text{ resol} \quad g_\infty^{-1}(C_{v,w} \cap H_\infty) \neq \emptyset\right)$

sketch of pf. $k^\times \curvearrowright H_\infty$ (induced by $\mathrm{Spec}\, k[[t]] \to H$ with $k^\times \curvearrowright \mathrm{Spec}\, k[[t]]$)

$k \curvearrowright$

$C_{v,w} \cap H_\infty \subset H_\infty$ is k-invariant.

$$\begin{tikzcd} H_\infty \arrow[r] & H \arrow[l, bend right, "s"'] \\ W_\infty \arrow[u, "g_\infty"] \arrow[r] & W \arrow[u, "g\ \text{resol}"'] \arrow[l, bend left, "s"] \end{tikzcd}$$

not nece. surj. $\to g_\infty$

$\rightsquigarrow 0 \cdot \alpha \in C_{v,w} \cap H_\infty$ $(\forall \alpha \in C_{v,w} \cap H_\infty)$ $\left(H_\infty \underset{\underset{s}{\longleftarrow}\ \text{section}}{\xrightarrow{\varphi_0}} H_0 \cong H\right)$

$0 \cdot \alpha \in s(H_0)$

$\hookrightarrow$ lifts to W_∞ $\rightsquigarrow g_\infty^{-1}(C_{v,w} \cap H_\infty) \neq \emptyset$.

Denef - Loeser's theory (arc spaces of quotient vars) [DL02]

motivation: McKay corresp (ch=0)

jet schm · arc space for $k[t]$-schm

- X : schm / $k[t]$ $\rightsquigarrow$ • $X_m \overset{\text{set}}{:=} \{ \operatorname{Spec} k[t]/t^{m+1} \to X : k[t]\text{-mor} \}$
  - $X_\infty := \varprojlim X_m \overset{\text{set}}{=} \{ \operatorname{Spec} k[[t]] \to X : k[t]\text{-mor} \}$
- cylinder & codim are defined similarly.

(• $X = \underbrace{Y}_{\text{var}/k} \times \operatorname{Spec} k[t] \rightsquigarrow X_m = Y_m$)

- ✡ "Change of variables formula" for $\underset{k[t]\text{-schm}}{X \to Y}$: $k[t]$-mor.
  (slide p7. Thm [DL99])

Arc space of quotient sing :
(want to study $(\bar{A}/G)_\infty$)

- $\overset{\text{fin}}{G} \subset GL_N(k)$, $G \curvearrowright \mathbb{A}^N_k =: \bar{A}$ free in codim 1
- $\gamma \in G$. $\gamma = \operatorname{diag}(\zeta^{e_1}, \cdots, \zeta^{e_N})$.
  ζ : d-th root of unity.
  $d := |G|$

$$
\begin{array}{ccccl}
k[t][x_1, \cdots, x_N]^G & \xrightarrow{\lambda_\gamma^*} & k[t][x_1, \cdots, x_N] & : & k[t]\text{-hom} \\
x_i & \longmapsto & t^{\frac{e_i}{d}} x_i & & \\
t & \longmapsto & t & &
\end{array}
$$

$(-)_\infty$ ⇓

$$(\bar{A}/G)_\infty \xleftarrow{\lambda_{\gamma\infty}} \bar{A}_\infty$$

$\coprod_{\gamma \in \operatorname{Conj} G}$ ⇓

$$(\bar{A}/G)_\infty \xleftarrow{\lambda_\infty} \coprod_{\gamma \in \operatorname{Conj} G} \bar{A}_\infty :$$

surj. finite. outside a thin set.

- codim of cylinder in $(\bar{A}/G)_\infty$ can be studied that in $\bar{A}_\infty$ by ✡ !

arc spaces of quotient vars:

$\underset{\text{fin}}{G} \curvearrowright \bar{A} := \mathbb{A}^N_k$, $\quad \gamma \in G$, $\gamma = \mathrm{diag}(\xi^{e_1}, \dots, \xi^{e_N})$

$\bar{A} \supset \bar{X}$: G-inv

(want to study $(\bar{X}/G)_\infty$)

$$\begin{tikzcd}
k[t][x_1,\dots,x_N]^G \arrow[r, "\lambda_\gamma^*"', "x_i \mapsto t^{\frac{e_i}{d}} x_i"] \arrow[d] & k[t][x_1,\dots,x_N] \arrow[d] \\
k[t][x_1,\dots,x_N]^G / I_{\bar{X}} \arrow[r] & k[t][x_1,\dots,x_N]/(\lambda_\gamma^*(I_{\bar{X}}))
\end{tikzcd}$$

Spec $\left\{ \begin{tikzcd}
\bar{A}/G \times \mathbb{A}^1 & \bar{A} \times \mathbb{A}^1 \arrow[l, "\lambda_\gamma"'] \\
\bar{X}/G \times \mathbb{A}^1 \arrow[u, phantom, "\cup"] & \bar{X}^{(\gamma)} \arrow[l] \arrow[u, phantom, "\cup"]
\end{tikzcd} \right.$ $\quad \bar{X}^{(\gamma)} := \operatorname{Spec}(\quad)$

$(-)_\infty \left\{ \begin{tikzcd}
(\bar{A}/G)_\infty & \bar{A}_\infty \arrow[l] \\
(\bar{X}/G)_\infty \arrow[u, phantom, "\cup"] & \bar{X}^{(\gamma)}_\infty \arrow[l] \arrow[u, phantom, "\cup"]
\end{tikzcd} \right.$

$\rightsquigarrow$ codim of cylinders in $(\bar{X}/G)_\infty$ can be studied by that in $\bar{X}^{(\gamma)}_\infty$.

eg. $X = (x_1^3 + x_2^3 + x_3^3 = 0) \subset \mathbb{A}^3$

$\gamma = \mathrm{diag}(1, \xi^1, \xi^2)$, $d = 3$

$\rightsquigarrow \bar{X}^{(\gamma)} = \operatorname{Spec} k[t][x_1, x_2, x_3]/(x_1^3 + t x_2^3 + t^2 x_3^3)$

$\gamma' = \mathrm{diag}(\xi^1, \xi^1, \xi^2)$

$\rightsquigarrow \bar{X}^{(\gamma')} = \operatorname{Spec} k[t][x_1, x_2, x_3]/t(x_1^3 + x_2^3 + t x_3^3)$

$\left(\bar{X}^{(\gamma)}_\infty = \{1\text{pt}\} \right)$

mld of hyperquot sing

$$G \curvearrowright \mathbb{A}^N = \bar{A} \qquad \bar{A}/G =: A$$

$$\cup \qquad\qquad\qquad \cup$$

$$\bar{X} = (f = 0) \leftarrow G\text{-inv.} \qquad \bar{X}/G =: X$$

$$\ni x = 0$$

suppose X : normal.

Thm [N - Shibata]

$$\operatorname{mld}_x(X, \mathfrak{a}^\delta) = \inf_{\substack{w, b \geq 0 \\ \gamma \in G}} \{ \operatorname{codim} C_{w,b,\gamma} + \operatorname{age}\gamma - b - \delta w \} \qquad \left(\operatorname{age}\gamma = \sum_i \frac{e_i}{d} \right)$$

$$C_{w,b,\gamma} = \left(\operatorname{Cont}^{\geq w}(\mathfrak{a}\mathcal{O}_{\bar{X}^{(\gamma)}}) \cap \operatorname{Cont}^{\geq 1}(\mathfrak{m}_x \mathcal{O}_{\bar{X}^{(\gamma)}}) \cap \operatorname{Cont}^{b}(\operatorname{Jac}_{\bar{X}^{(\gamma)}/k[t]}) \right) \subset \bar{X}^{(\gamma)}_\infty$$

sketch of proof

$$\operatorname{mld}_x(X, \mathfrak{a}^\delta) \overset{[\text{EMY}]}{=} \inf_{w, \ell \geq 0} \{ \operatorname{codim}(\quad) - \frac{\ell}{r} - \delta w \}$$

$$\left(\operatorname{Cont}^{\geq w}(\mathfrak{a}) \cap \operatorname{Cont}^{\ell}(\mathfrak{R}_{r,X}) \cap \operatorname{Cont}^{\geq 1}(\mathfrak{m}_x) \right) \subset X_\infty$$

$$\begin{array}{ccc} A_\infty & \xleftarrow{\lambda_{\gamma\infty}} & \bar{A}_\infty \\ \cup & & \cup \\ X_\infty & \xleftarrow[\mu_{\gamma\infty}]{} & \bar{X}^{(\gamma)}_\infty \end{array}$$

adjunction

$$\begin{array}{ccc} \mu_\gamma^* \omega_X & \xrightarrow{\operatorname{age}\gamma} & \omega_{\bar{X}^{(\gamma)}/k[t]} \\ \mathfrak{R}_{1,X} \to \ell \uparrow & & \uparrow \leftarrow \operatorname{Jac}_{\bar{X}^{(\gamma)}/k[t]} \ b \\ \mu_\gamma^* \Omega^n_X & \longrightarrow & \Omega^n_{\bar{X}^{(\gamma)}/k[t]} \end{array} , \qquad \lambda_\gamma^* \omega_A \xrightarrow{\operatorname{age}\gamma} \omega_{\bar{A}}$$

$\operatorname{Jac}_{\mu_\gamma}$ ← appeared in CVF.

$$\left(k[t^{\frac{1}{d}}][x_1, \dots, x_N]^G \xrightarrow{\lambda_\gamma^*} k[t^{\frac{1}{d}}][x_1, \dots, x_n] \right.$$

$$\searrow \qquad \nearrow$$

$$\text{crepant} \rightarrow k[t^{\frac{1}{d}}][x_1, \dots, x_n] \qquad t^{\frac{e_i}{d}} x_i \mapsfrom x_i \leftarrow \operatorname{age}\gamma$$

Proof of PIA for quotient sing

Thm [NS] Suppose X : klt.

$$\rightsquigarrow \operatorname{mld}_x(A, (f)\mathfrak{a}^\delta) = \operatorname{mld}_x(X, \mathfrak{a}|_X^\delta)$$

$(\geq ?)$

sketch of proof

$\exists_{w.v.\gamma}$ $\parallel$ Thm $\qquad\qquad$ $\wedge\!\!\mid$ Thm

$$\operatorname{codim}(\underbrace{C_{w.v.\gamma}}) + \operatorname{age}\gamma - \delta w - v \qquad \operatorname{codim}(D_{w.\ell.\gamma}) + \operatorname{age}\gamma - \ell - \delta w$$

$$\Big(\operatorname{Cont}^{\geq w}(\mathfrak{a}\mathcal{O}_{\overline{A}'}) \cap \operatorname{Cont}^{\geq v}(f\mathcal{O}_{\overline{A}'}) \cap \operatorname{Cont}^{\geq 1}(m_x\mathcal{O}_{\overline{A}'})\Big) \quad \Big(\operatorname{Cont}^{\geq w}(\mathfrak{a}\mathcal{O}_{\overline{X}^{(\gamma)}}) \cap \operatorname{Cont}^{\geq 1}(m_x\mathcal{O}_{\overline{X}^{(\gamma)}}) \cap \operatorname{Cont}^{\ell}\operatorname{Jac}_{\overline{X}^{(\gamma)}/k[t]}\Big)$$

$\overline{A}_\infty$? $\qquad\qquad\qquad$ $\overline{X}_\infty^{(\gamma)}$

can be compared by [EMY]'s technique.

necessary for EMY's argument.

- $C_{w.v.\gamma} \cap \overline{X}_\infty^{(\gamma)}$ is not a thin set.

$\Updownarrow$ Fact

$$\Big(\forall g : W \to \overline{X}_\infty^{(\gamma)} : \text{resol} \qquad g_\infty^{-1}(C_{w.v.\gamma} \cap \overline{X}_\infty^{(\gamma)}) \neq \emptyset\Big)$$

use klt assumption.

- $C_{w.v.\gamma} \cap \overline{X}^{(\gamma)}_\infty$ is not a thin set

pf. $\beta \in \overline{A}_\infty$: the trivial arc. ($\beta^* : k[t][x_1, \cdots, x_N] \to k[t]$, $x_i \mapsto 0$)

$\rightsquigarrow$ $\beta \in C_{w.v.\gamma} \cap \overline{X}^{(\gamma)}_\infty$

(considering $k^\times$-action)

$W \to \overline{X}^{(\gamma)}$: resolution

$$\{(0,\cdots,0)\} \times \mathbb{A}^1 = T \subset \overline{X}^{(\gamma)} \subset \mathbb{A}^N_{x_i} \times \mathbb{A}^1_t, \qquad \operatorname{Spec} k[[t]] \xrightarrow{\beta} T$$

ETS: β lifts to W_∞.

flat outside $t=0$. (since X : flat)

$$\begin{tikzcd} W \arrow[r, "\star"] & \overline{X}^{(\gamma)} \arrow[r] & \mathbb{A}^1 \\ & T \arrow[u, phantom, "\cup"] \arrow[ur, "\sim"'] & \end{tikzcd}$$

- $\star$ has rationally chain conn fibers outside $t=0$ by Hacon-McKernan.

- $\star$ has a section over T. by Graber-Harris-Starr.

$$\begin{tikzcd} W \arrow[r] & \overline{X}^{(\gamma)} \arrow[r] & \mathbb{A}^1 & \\ \tilde{T} \arrow[u, phantom, "\cup"] \arrow[r, "\sim"] & T \arrow[u, phantom, "\cup"] \arrow[ur, "\sim"] & \operatorname{Spec} k[[t]] \arrow[l, "\beta"'] \end{tikzcd}$$

$\rightsquigarrow$ β lifts to W_∞.

Fact Y : $k[t]$-var

$C \subset Y_\infty$: cylinder.

C : thin set

$\overset{\text{def}}{\iff}$ $\exists\, Z \subsetneq Y$ s.t. $C \subset Z_\infty$

$\iff$ $\forall (\exists)$ $\underset{\text{sm}}{W} \xrightarrow{g} Y$: resol, $g_\infty^{-1}(C) = \emptyset$

ACC for quotient sing

(cyclic quot.)

Thm (Reid-Tai type formula)

$$\underset{x \in X/G}{\mathrm{mld}}(\mathbb{A}^N/G, \mathfrak{a}) = \min_{\gamma \in G} \underset{x \in X/\langle\gamma\rangle}{\mathrm{mld}}(\mathbb{A}^N/\langle\gamma\rangle, \mathfrak{a}\mathcal{O}_{\mathbb{A}^N/\langle\gamma\rangle})$$

Cor (question in [Borisov97]) $N \geq 0$ fixed

$$\rightsquigarrow \{\mathrm{mld}_x(X) \mid X : \text{quot sing of dim } N\}$$
$$= \{ \quad " \quad \mid X : \text{cyclic} \quad " \quad \}$$

$\Leftarrow$ ACC by [Bor97], [Amb06]

toric

toric

toric w/ boundary