

Inversion of adjunction for quotient singularities.

2021/1/7
Fudan
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/ $\mathbb{R} = \overline{\mathbb{R}}$, $ch=0$.
 X : quot sing
 U
 H : \mathbb{P}^2 Cart div
 $x \in$

$$\left(\begin{array}{c} \uparrow \\ \text{mld}_x(x.H) = \text{mld}_x H \\ \text{---} \\ \uparrow \\ \text{invariant of sing} \\ \text{(in context of MMP)} \end{array} \right)$$

MMP: Termination of flips \Leftarrow still open

Shokurov introduced MLD.
proposed 2 conjs.

\circ ACC conj + LSC conj $\xRightarrow{\text{shokurov}}$ Termination of flips
 (Ascending chain cond) (lower semi-conti)
local prob global prob

\circ PIA conj: precise version of inversion of adjunction
 \mapsto useful for induction argument.
 (on dim)

ACC conj $d > 0, I \subset [0,1] : \text{DCC set} : \text{Fixed}$ there is no infinite decreasing seq.

$\Delta = \sum a_i D_i \in I \iff a_i \in I \forall i$ (def)

$\mapsto A(d, I) := \{ \text{mld}_x(X, \Delta) \mid \dim X = d, \Delta \in I, x \in X \}$

satisfies ACC.

LSC conj $(X, \Delta) : \text{log pair} \mapsto X \xrightarrow{\quad} \mathbb{R}_{\geq 0} \cup \{-\infty\}$ is lower semi-conti.

$x \longmapsto \text{mld}_x(X, \Delta)$

PIA conj $(X, \Delta) : \text{log pair}$

\cup
 $S : \text{normal Cart div}$
 \subseteq
 $x \quad \text{s.t. } S \not\subseteq \text{Supp } \Delta$

$\mapsto \text{mld}_x(X, \Delta + S) = \text{mld}_x(S, \Delta|_S)$

(cf.) BDD conj: $\forall d > 0 \exists a(d) \in \mathbb{R}$ s.t. $\forall_x X : d\text{-dim } \mathbb{Q}\text{-for } \text{mld}_x X \leq a(d)$.

(Known only for $d \leq 3$)

- ACC conj for $I = \emptyset \Rightarrow$ BDD conj.
- LSC conj \Rightarrow BDD conj $a(d) = d$.

(cf.) Thm (Shokurov) ACC conj + LSC conj \Rightarrow termination of flips.

Known results

Acc :

- ① $d \leq 2$: [Alexeev], [Shokurov]
- ② X : fixed (flex) var, I : finite : [Kawakita]
- ③ X : 3-dim sm, I : DCC \mapsto ACC on $[0,1]$: [Kawakita]
- ④ X : 3-dim canonical, I : finite : [N]
- ⑤ X : 3-dim, $I = \{0\} \mapsto$ ACC around $\mathbb{1}$: [Jiang]
($\mathbb{1}$ -gap theorem)

\leftarrow by classification of surf sing

\leftarrow application of ACC for LCT
(rationality of accump)

LSC :

- ① $d \leq 3$: [Ambro]
- ② X : smooth : [Ein, Mustata, Tasuda]
- ③ X : LCI : [Ein, Mustata]
- ④ X : quot sing : [N]

\leftarrow jet schm theory

\leftarrow Tasuda's twisted jet stack

PIA :

- ① $d \leq 3$:
- ② X : smooth : [EMT]
- ③ X : LCI : [EM]

\leftarrow jet schm theory

② \leftarrow ① + LSC④

$x \mapsto \underbrace{\text{mld}_x(S, \mathcal{O})}_{\text{mld}_x(X, \mathcal{O}_X(-s))} : \text{LSC?}$

$\text{mld}_x(X, \mathcal{O}_X(-s)) : \text{LSC by } \textcircled{4}$

Today :

- ① PIA for quot sing : $X : \mathbb{A}^N / \mathbb{G}_m$ - free in codim $\mathbb{1}$
 $\leftarrow G = \text{GL}_n(\mathbb{K})$, $S \subset X$: flex Cartier prime div
- ② LSC for hyper quot sing : (S, \mathcal{O})
- ③ Acc for quot sing (without boundary) : $A_{\text{quot}}(d) := \left\{ \text{mld}_x(X) \mid \begin{matrix} X = \text{quot sing} \\ \text{dim } X = d \end{matrix} \right\} : \text{Acc.}$

$\mathcal{O} : \mathbb{R}$ -ideal on X .

Idea of proof of \square : combining [EM⁰³] + [DL⁰²]

① Ein-Mustafa-Tasuda's proof for PIA for $\underbrace{H}_{d \cdot v} \subset \underbrace{X}_{sm}$

- describe $mld_x(H.a)$ in terms of arc spaces.
 - $mld_x(X.b)$ (codim of cylinders $\subset H_\infty.X_\infty$)
- ⊙ compare their descriptions.

↓
this method does not work when X : sing.

② Denef - Loeser's theory of the arc spaces of quotient vars

- $(Y/G)_\infty$ can be studied by W_∞ for some $\mathbb{R}[t]$ -val W .
- If $Y = \mathbb{A}^N$ \leadsto $W_\infty = (\mathbb{A}^N)_\infty$

\leadsto $mld_x(X.b)$ can be described in terms of $\underbrace{\mathbb{A}^N}_{sm}$.

$$\left(\begin{array}{l} H \subset X \\ \text{"} \quad \text{"} \\ S/G \subset \mathbb{A}^N/G \end{array} \right)$$

[EM⁰³]'s method works!

jet schm. arc space $X : \text{schm}/\mathbb{k}$

- \rightsquigarrow X_m : the m -th jet schm $\stackrel{\text{set}}{=} \{ \text{spec } \mathbb{k}[t]/(t^{m+1}) \rightarrow X \}$
- X_∞ : the arc space $\stackrel{\text{set}}{=} \{ \text{Spec } \mathbb{k}[[t]] \rightarrow X \}$
- $(m \geq n)$ $\Pi_{m,n} : X_m \rightarrow X_n$: truncation map. (induced by $\mathbb{k}[t]/t^{m+1} \rightarrow \mathbb{k}[t]/t^{n+1}$)
- $X_\infty \stackrel{\text{def}}{=} \varprojlim_m X_m$, $\varphi_m : X_\infty \rightarrow X_m$

Cylinder: $C \subset X_\infty \stackrel{\text{def}}{\iff} \exists m \geq 0 \exists \sum_m \subset X_m$ st. $C = \varphi_m^{-1}(S)$.
constructible subset

↓ typical example

Contact loci: $\mathfrak{a} \subset \mathcal{O}_X$: ideal

\rightsquigarrow $\left(\begin{array}{l} \bullet \text{Cont}^m(\mathfrak{a}) := \{ \gamma \in X_m \mid \text{ord}_\gamma \mathfrak{a} = m \} \\ \bullet \text{Cont}^{\geq m}(\mathfrak{a}) := \{ \quad \quad \quad \geq m \} \end{array} \right)$

$\left(\begin{array}{l} \text{Def} \\ \text{ord}_\gamma \mathfrak{a} := \sup \{ r \geq 0 \mid \gamma(\mathfrak{a}) \subset (t^r) \} \\ \text{if } Z = V(\mathfrak{a}) \subset X \\ (Z_{m-1} \subset X_{m-1}) \\ \text{closed} \end{array} \right) \left(= \varphi_{m-1}^{-1}(Z_{m-1}) \right)$

Codim of cylinder: $X : \text{var}/\mathbb{k}$, $C \subset X_\infty$: cylinder, $\text{Jac}_X := \text{Fitt}^n \Omega_X$ ($n := \dim X$)

- when $C \subset \text{Cont}^e(\text{Jac}_X)$: $\text{codim } C \stackrel{(m \gg 0)}{=} (m+1)\dim X - \dim \varphi_m(C)$
- general case: $\text{codim } C := \min_{e \geq 0} \text{codim}(C \cap \text{Cont}^e \text{Jac}_X)$ } well-def.

Fact: $\varphi_{m+1}(\text{Cont}^e(\text{Jac}_X)) \rightarrow \varphi_m(\text{Cont}^e(\text{Jac}_X))$ is piecewise trivial fibr w/ fiber $\cong \mathbb{A}^n$.

7 Ein-Mustafä - Yasuda's proof of PIA for sm var

Def $(R_{r,x})$ $\text{Im } \otimes = R_{r,x} \otimes_x (r k_x)$
 $(\Omega_x^n)^{\otimes r} \xrightarrow{\otimes} \underline{O_x(r k_x)}$
invertible

Thm [EMT] [EM] (X, α^δ) : log pair, $r k_x$: Cartier, $(r > 0)$

$$\left(\begin{aligned} \rightsquigarrow \text{mld}_x(X, \alpha^\delta) &= \inf_{w, l \geq 0} \{ \text{codim}(\text{cont}^w(\alpha) \cap \text{cont}^l(R_{r,x}) \cap \text{cont}^{\geq l}(m_x)) - \frac{l}{r} - \delta w \} \\ &= \inf_{w, l \geq 0} \{ \text{codim}(\text{cont}^{\geq w}(\alpha) \cap \text{cont}^l(R_{r,x}) \cap \text{cont}^{\geq l}(m_x)) - \frac{l}{r} - \delta w \} \end{aligned} \right)$$

NOTE: $R_{r,x} = O_x$ if $X = \text{sm}$
 $R_{r,x} = \text{Jac}_x$ if $X = \text{l.c.i.}$

cf.

Thm (Denef - Loeser)

$$\underbrace{X^-}_{\text{sm}} \xrightarrow{f} X : \text{proper birat mor}$$

$$\begin{array}{ccc} X^- & \xrightarrow{f_0} & X_\infty \\ \cup & & \cup \\ B & & A := f_\infty(B) \end{array}$$

cylinder

Def (Jac_f) $\text{Im } \otimes = \text{Jac}_f \otimes \Omega_x^n$
 $f^* \Omega_x^n \xrightarrow{\otimes} \underline{\Omega_{X^-}^n}$
inv

Suppose

$$B = \text{cont}^{e'} \text{Jac}_{X^-} \cap \text{cont}^e \text{Jac}_f$$

$$A = \text{cont}^{e''} \text{Jac}_X$$

for some $e, e', e'' \geq 0$

$$\rightsquigarrow \text{codim } B + e = \text{codim } A$$

$$\begin{array}{ccc} f^* \Omega_x^n & \xrightarrow{\text{Jac}_f} & \Omega_{X^-}^n \\ \downarrow \otimes r_x & & \downarrow \wr \end{array}$$

$$\begin{array}{ccc} f^* O(k_x) & \rightarrow & O(k_{X^-}) \\ \uparrow & & \\ \text{discrepancy} & & \end{array}$$

(\leq : always true)

$$\text{mld}_x(x, \mathcal{O}_x(-H)) \stackrel{?}{\geq} \text{mld}_x(H, \mathcal{O}_x(H))$$

want to show:
 setting $x \in H \subset X$
 hyper surf $\quad \quad \quad \text{sm}$

($\exists v, w$) || Thm

|| Thm

$$\text{codim}(C_{v,w}) - \delta w - v \stackrel{\text{want to compare}}{\leq} \text{codim}(D_{w,\mathcal{L}}) - l - \delta w$$

$$\left(\text{cont}^{\geq w}(C_{v,w}) \cap \text{cont}^{\geq v}(H) \cap \text{cont}^{\geq 1}(m_x) \right) \left(\text{cont}^{\geq w}(\mathcal{O}_H) \cap \text{cont}^{\geq l}(\text{Jac}_H) \cap \text{cont}^{\geq 1}(m_x) \right)$$

key claim

① If $C_{v,w} \cap H_{\infty} \cap \text{cont}^{\geq l}(\text{Jac}_H) \neq \emptyset \Rightarrow \text{codim}(C_{v,w} \cap H_{\infty}) \leq \text{codim} C_{v,w} + l - v$

② $\exists \mathcal{L}$ s.t. $C_{v,w} \cap H_{\infty} \cap \text{cont}^{\geq l}(\text{Jac}_H) \neq \emptyset$. $\left(\begin{array}{l} \Leftrightarrow W \xrightarrow{\mathcal{L}} H : \forall \text{ resol} \\ \text{Fact} \\ \text{sm} \end{array} \right) \left(\begin{array}{l} \mathcal{L}_{\infty}^{-1}(C_{v,w} \cap H_{\infty}) \neq \emptyset \end{array} \right)$

↑
 sketch of pf • $\mathbb{R}^x \curvearrowright H_{\infty}$ (induced by $\text{Spec } \mathbb{R}[[\epsilon]] \rightarrow H$)
 $\mathbb{R}^x \curvearrowright$

$C_{v,w} \cap H_{\infty} \subset H_{\infty}$
 is \mathbb{R} -invariant.
 $\mapsto \mathcal{O} \cdot \alpha \in C_{v,w} \cap H_{\infty} (\forall \alpha \in C_{v,w} \cap H_{\infty}) \left(\begin{array}{l} H_{\infty} \xrightarrow{\mathcal{L}_{\infty}} H_{\infty} \cong H \\ \uparrow \mathcal{L}_{\infty}^{-1} \text{ section} \end{array} \right)$

not nec. $\rightarrow \mathcal{L}_{\infty}^{-1}$ \uparrow $\mathcal{L}_{\infty}^{-1}$ \uparrow $\mathcal{L}_{\infty}^{-1}$ \uparrow $\mathcal{L}_{\infty}^{-1}$
 surj. $W_{\infty} \xrightarrow{\mathcal{L}_{\infty}^{-1}} W$

$\mathcal{O} \cdot \alpha \in S(H_{\infty})$
 \hookrightarrow lifts to $W_{\infty} \mapsto \mathcal{L}_{\infty}^{-1}(C_{v,w} \cap H_{\infty}) \neq \emptyset$.

motivation: McKay corresp (ch=0)

Denef - Loeser's theory (arc spaces of quotient vars) [DL02]

jet schm. arc space for $\mathbb{R}\llbracket t \rrbracket$ -schm

- $X : \text{schm} / \mathbb{R}\llbracket t \rrbracket \rightsquigarrow \bullet X_m := \left\{ \text{Spec } \mathbb{R}\llbracket t \rrbracket / t^{m+1} \rightarrow X : \mathbb{R}\llbracket t \rrbracket\text{-mon} \right\}$
- $X_\infty := \varprojlim X_m \stackrel{\text{set}}{=} \left\{ \text{Spec } \mathbb{R}\llbracket t \rrbracket \rightarrow X : \mathbb{R}\llbracket t \rrbracket\text{-mon} \right\}$

• cylinder & codim are defined similarly.

(• $X = \varprojlim_{\text{vac}/\mathbb{R}} X \times \text{Spec } \mathbb{R}\llbracket t \rrbracket \rightsquigarrow X_m = \varprojlim X_m$

★ "change of variables formula" for $X \xrightarrow{\mathbb{R}\llbracket t \rrbracket\text{-schm}} \gamma : \mathbb{R}\llbracket t \rrbracket\text{-mon}$.
(slide p7. Thu [DL99])

Arc space of quotient sing : $\circ \text{fin } GL_N(\mathbb{R})$
 (want to study $(\bar{A}/G)_\infty$) $\circ G \curvearrowright \mathbb{A}_{\mathbb{R}}^N =: \bar{A}$ • $\gamma \in G$. $\gamma = \text{diag}(\zeta^{\epsilon_1}, \dots, \zeta^{\epsilon_N})$.
 free in codim 1 $\zeta := d\text{-th root of unity}$.
 $d := |G|$

$$\mathbb{R}\llbracket t \rrbracket[x_1, \dots, x_N] \xrightarrow{\lambda_\gamma^*} \mathbb{R}\llbracket t \rrbracket[x_1, \dots, x_N] : \mathbb{R}\llbracket t \rrbracket\text{-hom}$$

$$x_i \mapsto t^{\frac{\epsilon_i}{d}} x_i$$

$$t \mapsto t$$

$(-)_\infty \}$

$$(\bar{A}/G)_\infty \xleftarrow{\lambda_{\gamma\infty}} \bar{A}_\infty$$

suiv. finite. outside a thin set.

$\bigsqcup_{\gamma \in \text{conj } G} \}$

$$(\bar{A}/G)_\infty \xleftarrow{\lambda_\infty} \bigsqcup_{\gamma \in \text{conj } G} \bar{A}_\infty$$

• codim of cylinder in $(\bar{A}/G)_\infty$ can be studied that in \bar{A}_∞ by ★!

arc spaces of quotient vars:

$$\begin{array}{c} \underbrace{G}_{\text{fin}} \curvearrowright \bar{A} := \mathbb{A}_{\mathbb{R}}^N \\ \cup \\ \bar{X} : G\text{-inv} \end{array} \quad \gamma \in G \\ \text{diag}(\xi^{e_1}, \dots, \xi^{e_N})$$

(want to study $(\bar{X}/G)_\infty$)

$$\mathbb{R}[t][x_1, \dots, x_N]^G \xrightarrow{\lambda_\gamma^*} \mathbb{R}[t][x_1, \dots, x_N]$$

$x_i \mapsto t^{a_i} x_i$

$$\mathbb{R}[t][x_1, \dots, x_N]^G / I_{\bar{X}} \longrightarrow \mathbb{R}[t][x_1, \dots, x_N] / (\lambda_\gamma^*(I_{\bar{X}}))$$

$$\text{Spec} \left\{ \begin{array}{ccc} \bar{A}/G \times A' & \xleftarrow{\lambda_\gamma} & \bar{A} \times A' \\ \cup & & \cup \\ \bar{X}/G \times A' & \longleftarrow & \bar{X}^{(\gamma)} := \text{Spec}(\quad) \end{array} \right\}$$

$$(-)_\infty \left\{ \begin{array}{ccc} (\bar{A}/G)_\infty & \longleftarrow & \bar{A}_\infty \\ \cup & & \cup \\ (\bar{X}/G)_\infty & \longleftarrow & \bar{X}_\infty^{(\gamma)} \end{array} \right.$$

\rightsquigarrow codim of cylinders in $(\bar{X}/G)_\infty$ can be studied by that in $\bar{X}_\infty^{(\gamma)}$.

eg. $X = (x_1^3 + x_2^3 + x_3^3 = 0) \subset \mathbb{A}^3$
 $\gamma = \text{diag}(1, \xi', \xi^2)$. $d = 3$

$\rightsquigarrow \bar{X}^{(\gamma)} = \text{Spec } \mathbb{R}[t][x_1, x_2, x_3] / (x_1^3 + tx_2^3 + t^2x_3^3)$

$\gamma' = \text{diag}(\xi', \xi', \xi^2)$

$\rightsquigarrow \bar{X}^{(\gamma')} = \text{Spec } \mathbb{R}[t][x_1, x_2, x_3] / t(x_1^3 + x_2^3 + tx_3^3)$

$(\bar{X}_\infty^{(\gamma')} = \{1, \pi\})$

mld of hyperquot sing

$$G \curvearrowright \mathbb{A}^n = \bar{A} \\ \downarrow \subset \\ X = (f=0) \\ \downarrow \subset \\ x=0$$

G -inv.

$$\bar{A}/G =: A \\ \cup \\ \bar{X}/G =: X$$

suppose X : normal.

Thm [N. Shibata]

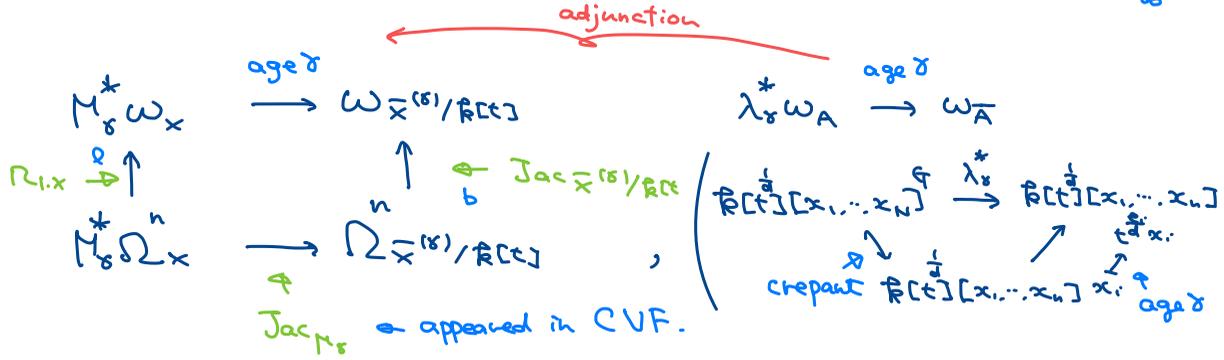
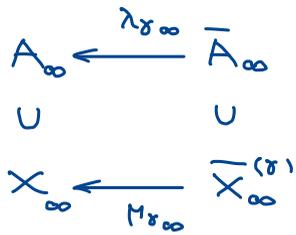
$$\left(\text{mld}_x(X, \alpha^\delta) = \inf_{\substack{w, b \geq 0 \\ \delta \in G}} \left\{ \text{codim } \underbrace{C_{w,b,r}}_{!!} + \text{age } \delta - b - \delta w \right\} \right) \quad \left(\text{age } \delta = \sum_i \frac{a_i}{q_i} \right)$$

$$\left(\text{cont}^{\geq w}(\alpha O_{\bar{X}}^{(s)}) \cap \text{cont}^{\geq 1}(m_x O_{\bar{X}}^{(s)}) \cap \text{cont}^b(\text{Jac}_{\bar{X}}^{(s)}/\mathbb{R}[t]) \right) \subset \bar{X}_\infty^{(s)}$$

sketch of proof

$$\text{mld}_x(X, \alpha^\delta) \stackrel{[EMF]}{=} \inf_{w, \ell \geq 0} \left\{ \text{codim}(\uparrow) - \frac{\ell}{r} - \delta w \right\}$$

$$\left(\text{cont}^{\geq w}(a) \cap \text{cont}^\ell(r, r, x) \cap \text{cont}^{\geq 1}(m_x) \right) \subset X_\infty$$



Proof of PIA for quotient sing

Thm [NS] Suppose $X : \mathbb{R}^2$.

$$\rightsquigarrow \text{mld}_x(A, (f) \alpha^\delta) = \text{mld}_x(X, \alpha|_X^\delta)$$

(\geq ?)

Sketch of proof

$\exists w, v, \delta \parallel \text{Thm}$

$$\text{codim}(C_{w, v, \delta}) + \text{age } \delta - \delta w \cdot v$$

$\forall \text{Thm}$

$$\text{codim}(D_{w, v, \delta}) + \text{age } \delta - \delta w - \delta v$$

$$\left(\text{cont}^{\geq w}(\alpha|_{\overline{A}}) \cap \text{cont}^{\geq v}(f|_{\overline{A}}) \cap \text{cont}^{\geq 1}(m_x|_{\overline{A}}) \right) \left(\text{cont}^{\geq w}(\alpha|_{\overline{X}}) \cap \text{cont}^{\geq 1}(m_x|_{\overline{X}}) \cap \text{cont}^{\geq 2} \text{Jac}_{\overline{X}/\mathbb{R}} \right)$$

$\overline{A} \quad \uparrow \quad \uparrow \quad \overline{X}$

can be compared by [EMT]'s technique.

- $C_{w, v, \delta} \cap \overline{X}_\infty^{(r)}$ is not a thin set.

necessary for EMT's argument.

\Updownarrow Fact

$$\left(\forall g: W \rightarrow \overline{X}_\infty^{(r)} : \text{resol} \quad g_\infty^{-1}(C_{w, v, \delta} \cap \overline{X}_\infty^{(r)}) \neq \emptyset \right)$$

\uparrow use \mathbb{R}^2 assumption.

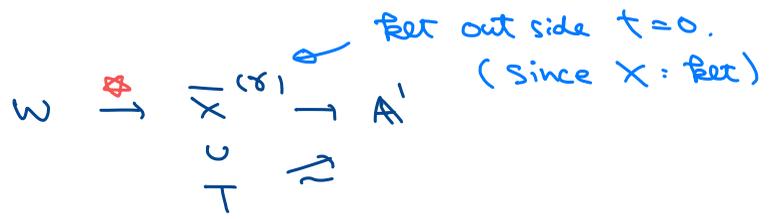
- $C_{w.v.s} \cap \overline{X}_0^{(18)}$ is not a thin set

pf. $\beta \in \overline{A}_\infty$: the trivial arc. $(\beta^*: \mathbb{R}[t][[x_1, \dots, x_n]] \rightarrow \mathbb{R}[t])$
 $x_i \mapsto 0$
 $\mapsto \beta \in C_{w.v.s} \cap \overline{X}_0^{(18)}$

(considering \mathbb{R}^x -action)

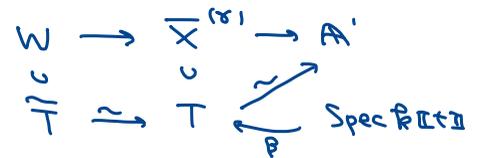
$W \rightarrow \overline{X}^{(18)}$: resolution

ETS: β lifts to W_∞ .

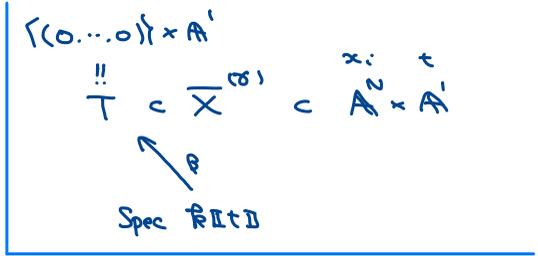


- \star has rationally chain conn fibers outside $t=0$ by Hacon-McKernan.

- \star has a section over T . by Graber-Harris-Starr.



$\mapsto \beta$ lifts to W_∞ .



Fact $Y: \mathbb{R}[t]$ -var

$C \subset Y_0$: cylinder.

C : thin set

\Leftrightarrow def $\exists Z \subset Y$ s.t. $C \subset Z_0$

$\Leftrightarrow \underbrace{W}_{sm} \xrightarrow{\beta} Y$: resol $g_0^{-1}(C) = \emptyset$

ACC for quotient sing.

(cycliz quot.)

Thm (Reid-Tai type formula)

$$mld_x(\mathbb{A}^n/G, \alpha) = \min_{\gamma \in G} mld_x(\mathbb{A}^n/\langle \gamma \rangle, \alpha \circ \mathbb{A}^n/\langle \gamma \rangle)$$

Cor (question in [Bor97]) $N \geq 0$ fixed

$$\rightsquigarrow \left\{ mld_x(x) \mid x: \text{quot sing of dim } N \right\} \\ = \left\{ \quad \mid x: \text{cycliz} \quad \right\}$$

Acc by [Bor97], [Amb06] toric toric w/ boundary