Descendents for stable pairs on 3-folds

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Idea of descendents

GW theory / X nonsingular projective Variety

$$
\begin{tikzcd}
 & \overline{M}_{g,1}(X,\beta) \arrow[dl, "\pi"'] \arrow[dr, "ev"] & \\
\overline{M}_g(X,\beta) & & X
\end{tikzcd}
$$

$$\widetilde{\tau}_k(\gamma) \in H^*(\overline{M}_g(X,\beta))$$

Nonstandard definition!

$$\gamma \in H^*(X)$$

$$\widetilde{\tau}_k(\gamma) = \pi_*\left(\psi_1^k \cup ev^*(\gamma)\right)$$

ψ_1 is the cotangent line class

The study of descendent classes in GW theory is a long story

- Tautological ring of $\overline{M}_{g,n}$

  Pixton's relations

- Witten's Conjecture

  Kontsevich, OP, Mirzakhani

- Virasoro constraints

  Eguchi, Hori, Xiong

most relevant direction today

The topic today is descendent classes on moduli spaces of sheaves

- dim 1

$\mathcal{U}_{C,2,\mathcal{L}}$ = moduli space of rank 2 stable bundles on a fixed curve C of genus g with a fixed det $= \mathcal{L}$.

$\boxed{\deg \mathcal{L} = 1}$

Universal bundle

$$\begin{tikzcd}
 & \mathcal{E} \arrow[d] & \\
 & \mathcal{U} \times C \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\mathcal{U} & & C
\end{tikzcd}$$

$$\tau_k(\gamma) = \pi_{1*}\left(ch_k(\mathcal{E}) \cup \pi_2^*(\gamma)\right)$$

$\gamma \in H^*(C)$

Theorem: $H^*(\mathcal{U})$ is generated by descendents.

Mumford, Kirwan, Zagier, ... (also find relations).

- dim 2

Moduli of sheaves on Surfaces

Descendent theory started with Donaldson.

A more recent example: Quot schemes on Surfaces

Let S be a nonsingular projective surface

$\mathrm{Quot}_S(\mathbb{C}^r, \beta, n)$ of quotients

$$\mathbb{C}^r \otimes \mathcal{O}_S \to Q \to 0$$

where

$$\mathrm{rk}\, Q = 0$$
$$c_1(Q) = \beta$$
$$\chi(Q) = n$$

deformation theory $\qquad 0 \to S \to \mathbb{C}^r \to Q \to 0$

def $\qquad \mathrm{Ext}^0(S, Q)$

Obs $\qquad \mathrm{Ext}^1(S, Q)$

higher $\qquad \mathrm{Ext}^2(S, Q) = \mathrm{Ext}^0(Q, S \otimes k)^\vee = 0$

since Q is torsion

so we have a 2-term obstruction theory and a virtual fundamental class

$$[\mathrm{Quot}_S(\mathbb{C}^r, \beta, n)]^{vir}$$ of dim $r \cdot n + \beta^2$

Marian - Oprea - P 2018

Oprea - P 2019

$$
\begin{tikzcd}
 & \mathcal{F} \arrow[d] & \text{universal quotient} \\
 & \mathrm{Quot}(\mathbb{C}^r, \beta, n) \times S \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\mathrm{Quot} & & S
\end{tikzcd}
$$

$$\tau_k(\gamma) = \pi_{1*}\left(\mathrm{ch}_k \mathcal{F} \cdot \pi_2^*(\gamma) \right)$$

$$\gamma \in H^*(S)$$

descendent series

$$\left\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_\ell}(\gamma_\ell) \right\rangle^S_{r,\beta} = \sum_{n \in \mathbb{Z}} q^n \int_{[\mathrm{Quot}(r,\beta,n)]^{vir}} \tau_k(\gamma_1) \cdots \tau_{k_\ell}(\gamma_\ell)\, c(T^{vir}) \in \mathbb{Q}((q))$$

Conj [D. Johnson, Oprea, P 2020]

The series is Laurent expansion of a rational function in q.

The virtual geometry of

$\mathrm{Quot}_S(\mathbb{C}^r, \beta, n)$ [MOP]

Nekrasov partition function

[Arbesfeld, ...]

K theory

Donaldson/SW invariants

[OP, Woonam Lim]

see also Duerr-Kabanov-Okonek

χ_y-genus

Lehn's Conj about Segre integrals $\mathrm{Hilb}(S, n)$

$S = K3$

Voisin

Ex. S is a minimal surface of general type with a nonsing canonical curve of genus 3

$$\langle 1 \rangle^S_{2, K_S} = (-1)^{\chi(\mathcal{O}_S)} \cdot \frac{(128q^4 - 64q^3 + 8q^2 - 16q + 8)}{q\,(1-4q)^4}$$

SW invariant

- dim 3

The main topic of the lecture concerns the moduli of stable pairs on 3-folds

X is nonsingular proj 3-fold

$\beta \in H_2(X, \mathbb{Z})$ curve class

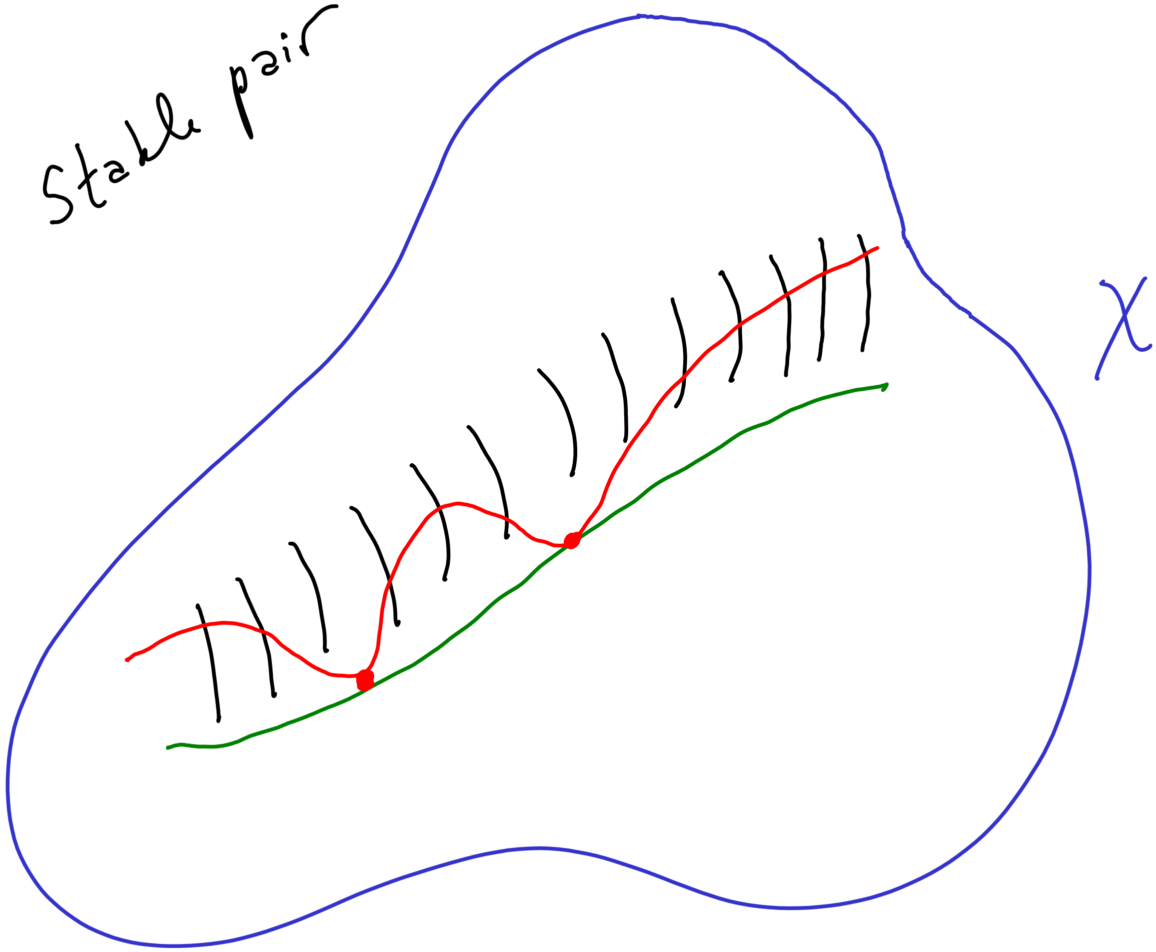
$n \in \mathbb{Z}$

$P_n(X, \beta)$ is the moduli of stable pairs

$[\mathcal{F}, s] \in P_n(X, \beta)$

- $\mathcal{F}$ is pure sheaf of dim 1.
- $\mathcal{O}_X \xrightarrow{s} \mathcal{F}$ is a section with coker of dim 0

Stable pair



s section

$\mathcal{F}$ sheaf $n = \chi(\mathcal{F})$

$\mathrm{Supp}(\mathcal{F})$ $\beta = [\mathrm{Supp}(\mathcal{F})]$

Le Potier, a series of papers P–Thomas

Ex. $X = \mathbb{P}^3$

Then $P_n(X, d) \supset$ Classical locus

parameterizing:

$C \subset \mathbb{P}^3$ nonsing irreducible curve of degree d

$F \to C$ a line bundle of degree ℓ

$s \in H^0(C, F)$ a nonzero section

here $n = \ell - \text{genus}(C) + 1$

Note: $P_n(X, d)$ contains more degenerate objects if we consider $\mathbb{I} = [\,\mathcal{O}_X \xrightarrow{s} F\,]$ as an object in $D^b_{Coh}(X)$, then

def $\quad \mathrm{Ext}^1_0(\mathbb{I}, \mathbb{I})$

Obs $\quad \mathrm{Ext}^2_0(\mathbb{I}, \mathbb{I})$

higher vanish

So we have a virtual fundamental class

$$[\,P_n(X, \beta)\,]^{vir} \quad \text{of} \quad \dim \int_\beta c_1(X)$$

See my paper with R. Thomas "Counting curves via stable pairs..."

Descendents

$$\begin{tikzcd} & \mathbb{F} \arrow[d] & \\ & P_n(X,\beta) \times X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ P_n(X,\beta) & & X \end{tikzcd}$$

universal sheaf

Definition:

$$\tau_k(\gamma) = \pi_{1*}\left(ch_{k+2}(\mathbb{F}) \cup \pi_2^*(\gamma) \right)$$

$\gamma \in H^*(X)$

$-\mathbb{I}$

Better convention

$$ch_k(\gamma) = \pi_{1*}\left(ch_k(\mathbb{F} - \mathcal{O}) \cup \pi_2^*(\gamma) \right)$$

Conjectures

GW/Pairs Correspondence

MNOP ~2005
Pixton-P ~2012
OOP 2019

Rationality

MNOP ~2005
Pixton-P ~2012

Virasoro

OOP 2019
Moreira OOP
Moreira 2020 (hopefully)

M = Maulik
N = Nekrasov
O = Okounkov
OO = Oblomkov Okounkov

Rationality

$$\left\langle \mathrm{ch}_{k_1}(\gamma_1) \cdots \mathrm{ch}_{k_r}(\gamma_r) \right\rangle_\beta^X = \sum_{n \in \mathbb{Z}} q^n \int_{[P_n(X,\beta)]^{vir}} \mathrm{ch}_{k_1}(\gamma_1) \cdots \mathrm{ch}_{k_r}(\gamma_r) \in \mathbb{Q}((q))$$

Conj (Rationality)

$\left\langle \mathrm{ch}_{k_1}(\gamma_1) \cdots \mathrm{ch}_{k_r}(\gamma_r) \right\rangle_\beta^X$ is the Laurent expansion of a rational function in q.

Virasoro

Universal relations among descendent series

X is a nonsingular proj 3-fold
with only (p,p) cohomology

Main example: X is a toric 3-fold

Moreira treats the general case

Definition: Let $\mathbb{D}^X$ be
the commutative $\mathbb{Q}$-algebra
with generators $\{ ch_i(\gamma) \mid i \geq 0,\ \gamma \in H^*(X,\mathbb{Q}) \}$

subject to the basic relations

$$\mathrm{ch}_i(\lambda \cdot \gamma) = \lambda \, \mathrm{ch}_i(\gamma) \qquad \lambda \in \mathbb{Q}$$

$$\mathrm{ch}_i(\gamma + \hat{\gamma}) = \mathrm{ch}_i(\gamma) + \mathrm{ch}_i(\hat{\gamma}) \qquad \gamma, \hat{\gamma} \in H^*(X).$$

In order to define the Virasoro Constraints, we require 3 constructions in $\mathbb{D}^X$

(i) Define the derivation for $k \geq -1$

$$R_k : \mathbb{D}^X \to \mathbb{D}^X \quad \text{by}$$

$$R_k(\mathrm{ch}_i(\gamma)) = \left(\prod_{n=0}^{k} (i+d-3+n) \right) \mathrm{ch}_{i+k}(\gamma)$$

for R_{-1} we have $R_{-1}(\mathrm{ch}_i(\gamma)) = \mathrm{ch}_{i-1}(\gamma)$.

(ii) Define $\mathrm{ch}_a \mathrm{ch}_b(\gamma) \in \mathbb{D}^X$ by

$$\mathrm{ch}_a \mathrm{ch}_b(\gamma) = \sum_i \mathrm{ch}_a(\gamma_i^L)\, \mathrm{ch}_b(\gamma_i^R)$$

where $\sum_i \gamma_i^L \otimes \gamma_i^R$ is the Künneth decomposition of $\gamma \cdot \Delta \in H^*(X \times X)$.

The notation

$$(-1)^{d^L d^R} (a + d^L - 3)!\,(b + d^R - 3)!\; \mathrm{ch}_a \mathrm{ch}_b(\gamma)$$

will be used for

$$\sum_i (-1)^{d(\gamma_i^L) d(\gamma_i^R)} (a + d(\gamma_i^L) - 3)!\,(b + d(\gamma_i^R) - 3)! \cdot \mathrm{ch}_a(\gamma_i^L)\, \mathrm{ch}_b(\gamma_i^R)$$

$d(\ )$ denotes the complex degree!

(iii) define operator $T_k : \mathbb{D}^X \to \mathbb{D}^X$ by

$$T_k = -\frac{1}{2} \sum_{a+b=k+2} (-1)^{d^L d^R} (a+d^L-3)!\,(b+d^R-3)!\; ch_a ch_b (c_1)$$

$$+ \frac{1}{24} \sum_{a+b=k} a!\, b!\; ch_a ch_b (c_1 c_2)$$

- in sums $a, b \geq 0$
- factorials with negative arguments vanish
- $c_1, c_2 \in H^*(X)$ are the Chern classes of $\mathrm{Tan}(X)$.

Define the constraint operator

$$\mathcal{L}_k = T_k + R_k + (k+1)!\, R_{-1}\, \mathrm{ch}_{k+1}(p)$$

for $k \geq -1$

Should view the point class here as $p = \frac{c_1 c_2}{24}$

Virasoro Conj [Moreira OOP]

X has only (p,p) cohomology

$\beta \in H_2(X, \mathbb{Z})$ curve class

$D \in \mathbb{D}^X$ any element.

Then $$\left\langle \mathcal{L}_k (D) \right\rangle_\beta^X = 0$$

for $k \geq -1$.

Ex. $\mathcal{X} = \mathbb{P}^3$

$$\mathcal{L}_1(D) = \left(-4\,ch_3(H) + R_1 + 2\,ch_2(p)R_{-1}\right) D$$

Try $D = ch_3(p)$ and $\beta = L$

$$\Downarrow$$

$$-4\langle ch_3(H)\,ch_3(p)\rangle_L^{\mathbb{P}^3} + 12\langle ch_4(p)\rangle_L^{\mathbb{P}^3} +$$

$$+ 2\langle ch_2(p)\,ch_2(p)\rangle_L^{\mathbb{P}^3} = 0$$

Check

$$\frac{3}{4}q - \frac{3}{2}q^2 + \frac{3}{4}q^3$$

$$\frac{1}{12}q - \frac{5}{6}q^2 + \frac{1}{12}q^3$$

$$q + 2q^2 + q^3$$

<u>Thrm</u> (Moreira OOP 2020) Let X be toric 3-fold.

for all $D \in \mathbb{D}^X_+$,

the Virasoro Constraints hold

$$\left\langle \mathcal{L}_k(D) \right\rangle^X_\beta = 0.$$

Def: $\mathbb{D}^X_+ \subset \mathbb{D}^X$

generated by $\{ ch_i(\gamma) \mid i \geq 0,\ \gamma \in H^{>0}(X) \}$.

Stationary descendents

Path of Proof

X is a nonsingular proj toric 3 fold

GW Virasoro Constraints hold

Semisimple / Givental–Teleman theory

~ 2010

↓

Lose control of descendents of 1 here

GW/Pairs descendent Correspondence

MNOP ~2005

Pixton–P ~2012

OOP 2019

(using older papers on local curves).

Formula in Stationary Case

↓

Transfer Virasoro Constraints from GW to Pairs in Stationary Case

Actually, I would like run the whole argument in the other direction.

Challenge: Prove Virasoro constraints for stable pairs on the moduli of sheaves!

Challenge: Control the descendents of $1 \in H^*(X)$.

Comments on the Conjectural functional equation

X nonsingular proj 3-fold

Rationality:

$$\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_r}(\gamma_r) \rangle_\beta^X \quad \text{is a rational function of } q$$

Functional equation:

q dependence

$$d_\beta = \int_\beta c_1(X)$$

$$\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_r}(\gamma_r) \mid q \rangle_\beta^X =$$

$$(-1)^{\sum_i k_i} q^{-d_\beta} \langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_r}(\gamma_r) \mid \tfrac{1}{q} \rangle_\beta^X$$

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subject to the natural relations

$$\begin{aligned}\tau_i(\lambda \cdot \gamma) &= \lambda\, \tau_i(\gamma)\,,\\ \tau_i(\gamma + \widehat{\gamma}) &= \tau_i(\gamma) + \tau_i(\widehat{\gamma})\end{aligned}$$

for $\lambda \in \mathbb{Q}$ and $\gamma, \widehat{\gamma} \in H^*(X)$. The subalgebra $\mathbb{D}^{X+}_{\mathrm{GW}} \subset \mathbb{D}^{X}_{\mathrm{GW}}$ of stationary descendents is generated by

$$\left\{ \tau_i(\gamma) \,\middle|\, i \geqslant -2\,, \gamma \in H^{>0}(X, \mathbb{Q}) \right\}.$$

We will use Getzler's renormalization $\mathfrak{a}_k$ of the Gromov-Witten descendents[6]:

$$\sum_{n=-\infty}^{\infty} z^n \tau_n = \mathsf{Z}^0 + \sum_{n>0} \frac{(\imath u z)^{n-1}}{(1+zc_1)_n} \mathfrak{a}_n + \frac{1}{c_1} \sum_{n<0} \frac{(\imath u z)^{n-1}}{(1+zc_1)_n} \mathfrak{a}_n\,, \tag{7}$$

$$\mathsf{Z}^0 = \frac{z^{-2}u^{-2}}{\mathcal{S}\left(\frac{zu}{\theta}\right)} - z^{-2}u^{-2},$$

where we use standard notation for a Pochhammer symbol

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}\,.$$

For example[7],

$$\tau_0(\gamma) = \mathfrak{a}_1(\gamma) + \frac{1}{24}\int_X \gamma c_2\,, \tag{8}$$

$$\tau_1(\gamma) = \frac{\imath u}{2}\mathfrak{a}_2(\gamma) - \mathfrak{a}_1(\gamma \cdot c_1)\,. \tag{9}$$

For $k \geqslant 2$ and $\gamma \in H^{>0}(X)$, we have the general formula

$$\tau_k(\gamma) = \frac{(\imath u)^k}{(k+1)!}\mathfrak{a}_{k+1}(\gamma) - \frac{(\imath u)^{k-1}}{k!}\left(\sum_{i=1}^{k}\frac{1}{i}\right)\mathfrak{a}_k(\gamma \cdot c_1) + \frac{(\imath u)^{k-2}}{(k-1)!}\left(\sum_{i=1}^{k-1}\frac{1}{i^2} + \sum_{1 \leqslant i < j \leqslant k-1}\frac{1}{ij}\right)\mathfrak{a}_{k-1}(\gamma \cdot c_1^2)\,. \tag{10}$$

For the special cases $k=-1$ and $k=-2$, we have

$$\tau_{-1}(\gamma) = (\imath u)^{-2}\mathfrak{a}_{-1}(\gamma) + (\imath u)^{-3}\mathfrak{a}_{-2}(c_1 \cdot \gamma), \quad \tau_{-2}(\gamma) = -(\imath u)^{-3}\mathfrak{a}_{-2}(\gamma).$$

[6]We use $\imath$ for the square root of -1. The genus variable u will usually occur together with $\imath$.

[7]The constant term $\frac{1}{24}\int_X \gamma c_2$ in the formula does not contribute unless $\gamma \in H^2(X)$.

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0.6. **Stationary** GW/PT **correspondence.** We define the subalgebras

$$\mathbb{D}^X_{\mathrm{GW}\circ} \subset \mathbb{D}^X_{\mathrm{GW}} \quad \text{and} \quad \mathbb{D}^{X+}_{\mathrm{GW}\circ} \subset \mathbb{D}^{X+}_{\mathrm{GW}}$$

by requiring the descendent indices for the generators to be non-negative: $\mathbb{D}^X_{\mathrm{GW}\circ}$ is generated by

$$\{\, \tau_i(\gamma) \,|\, i \geqslant 0\,, \gamma \in H^*(X) \,\}$$

and $\mathbb{D}^{X+}_{\mathrm{GW}\circ}$ is generated by

$$\{\, \tau_i(\gamma) \,|\, i \geqslant 0\,, \gamma \in H^{>0}(X) \,\}\,.$$

The stationary GW/PT transformation is the linear map

$$\mathfrak{C}^\bullet : \mathbb{D}^{X+}_{\mathrm{PT}} \to \mathbb{D}^{X+}_{\mathrm{GW}}$$

satisfying

$$\mathfrak{C}^\bullet(1) = 1$$

and defined on monomials by

$$\mathfrak{C}^\bullet\left(\widetilde{\mathsf{ch}}_{k_1}(\gamma_1)\ldots\widetilde{\mathsf{ch}}_{k_m}(\gamma_m)\right) = \sum_{P \text{ set partition of } \{1,\ldots,m\}} \prod_{S\in P} \mathfrak{C}^\circ\left(\prod_{i\in S}\widetilde{\mathsf{ch}}_{k_i}(\gamma_i)\right),$$

where the operations $\mathfrak{C}^\circ$ are

$$(11)\quad \mathfrak{C}^\circ\left(\widetilde{\mathsf{ch}}_{k+2}(\gamma)\right) = \frac{1}{(k+1)!}\mathfrak{a}_{k+1}(\gamma) + \frac{(\imath u)^{-1}}{k!}\sum_{|\mu|=k-1}\frac{\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}(\gamma\cdot c_1)}{\mathrm{Aut}(\mu)}$$
$$+ \frac{(\imath u)^{-2}}{k!}\sum_{|\mu|=k-2}\frac{\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}(\gamma\cdot c_1^2)}{\mathrm{Aut}(\mu)} + \frac{(\imath u)^{-2}}{(k-1)!}\sum_{|\mu|=k-3}\frac{\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}\mathfrak{a}_{\mu_3}(\gamma\cdot c_1^2)}{\mathrm{Aut}(\mu)},$$

$$(12)\quad \mathfrak{C}^\circ\left(\widetilde{\mathsf{ch}}_{k_1+2}(\gamma)\widetilde{\mathsf{ch}}_{k_2+2}(\gamma')\right) = -\frac{(\imath u)^{-1}}{k_1!k_2!}\mathfrak{a}_{k_1+k_2}(\gamma\gamma') - \frac{(\imath u)^{-2}}{k_1!k_2!}\mathfrak{a}_{k_1+k_2-1}(\gamma\gamma'\cdot c_1)$$
$$- \frac{(\imath u)^{-2}}{k_1!k_2!}\sum_{|\mu|=k_1+k_2-2}\max(\max(k_1,k_2),\max(\mu_1+1,\mu_2+1))\frac{\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}}{\mathrm{Aut}(\mu)}(\gamma\gamma'\cdot c_1)\,,$$

$$(13)\quad \mathfrak{C}^\circ\left(\widetilde{\mathsf{ch}}_{k_1+2}(\gamma)\widetilde{\mathsf{ch}}_{k_2+2}(\gamma')\widetilde{\mathsf{ch}}_{k_3+2}(\gamma'')\right) = \frac{(\imath u)^{-2}|k|}{k_1!k_2!k_3!}\mathfrak{a}_{|k|-1}(\gamma\gamma'\gamma''),\quad |k| = k_1+k_2+k_3\,.$$

In equations (11)-(13) above, the parts of μ are *positive* integers,

$$k\,,\ k_i \geqslant 0\,,$$

and all ocurrences of $\mathfrak{a}_0$ and $\mathfrak{a}_{-1}$ are set to 0 except in certain cases for lower values of k and k_i which are described below.[8] Because of the stationary assumption, all higher $\mathfrak{C}^\circ$ operations vanish. We will reintroduce these formulas in Sections 3 and 5 at the first instance of their application.

[8]The corrections for the lower vaules are consistent with the rule $\mathfrak{a}_0 = 0$, $\mathfrak{a}_{-1}/(-1)! = -u^2\tau_{-2}$.

The End