$$\mathcal{H} \circlearrowleft GL_2(\mathbb{R})^+$$

$$\beta \in GL_2(\mathbb{R})^+, \quad k \in \mathbb{Z}$$

$$f : \mathcal{H} \to \mathbb{C}$$

Notation:

$$f[\beta]_k : \mathcal{H} \to \mathbb{C}.$$

$$(f[\beta]_k)(\tau) = (\det\beta)^{k-1} j(\beta,\tau)^{-k} f(\beta(\tau))$$

where $j(\beta,\tau) = c\tau + d.$

$$\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

f is weakly modular of wt k iff f is meromorphic satisfying

$$f[\beta]_k = f \quad \forall \beta \in SL_2(\mathbb{Z})$$

$\Gamma \subset SL_2(\mathbb{Z})$ congruent subgp.

f weakly mod of wt k level Γ. $f[\beta]_k = f$ $\forall \beta \in \Gamma$.

Def: f is a cusp form of wt k level Γ if $f[\alpha]_k$ vanishes at $i\infty$ for every $\alpha \in SL_2(\mathbb{Z})$.

Hecke operator

Γ_1, Γ_2 congruence subgps.

$\alpha \in GL_2(\mathbb{Q})^+$

Lem: $\alpha^{-1}\Gamma_1\alpha \cap \Gamma_2 \in \Gamma_2$ is still a congruence subgp.

$\Gamma_1 \backslash \Gamma_1 \alpha \Gamma_2$ is finite.

$$\Gamma_1 \alpha \Gamma_2 = \bigsqcup_j \Gamma_1 \beta_j$$

$\beta_j \in \Gamma_1 \alpha \Gamma_2$. finite disjoint union.

$[\Gamma_1 \alpha \Gamma_2]_k$: operator

$f \in M_k(\Gamma_1)$

$$(f[\Gamma_1 \alpha \Gamma_2]_k)(\tau) := \sum_j f[\beta_j]_k$$

<u>Rmk</u>: (1) $f[\beta_j']_k = f[\beta_j]_k$ if $\Gamma_1 \beta_j' = \Gamma_1 \beta_j$

(2) For $\gamma \in \Gamma_2$

$$\left(\left(f[\Gamma_1 \alpha \Gamma_2]_k\right)[\gamma]_k\right)(\tau)$$

$$= \sum_j \left(f[\beta_j]_k[\gamma]_k\right)(\tau)$$

$$= \sum_j \left(f[\beta_j \gamma]_k\right)(\tau)$$

If $\beta_1, \ldots, \beta_s$ is a set of repr of $\Gamma_1 \alpha \Gamma_2$.

Then, $\beta_1\gamma, \ldots, \beta_s\gamma$ is a set repr of $\Gamma_1 \alpha \Gamma_2$

$$\Longrightarrow \left(f[\Gamma_1 \alpha \Gamma_2]_k\right)[\gamma]_k = f[\Gamma_1 \alpha \Gamma_2]_k$$

Not hard to see

$$[\Gamma_1 \alpha \Gamma_2]_k : \mathcal{M}_k(\Gamma_1) \to \mathcal{M}_k(\Gamma_2)$$

$$\hspace{4em} \cup\!| \hspace{6em} \cup\!|$$

$$\mathcal{S}_k(\Gamma_1) \to \mathcal{S}_k(\Gamma_2)$$

$$\Gamma_1 \backslash GL_2(\mathbb{Q})^+ / \Gamma_2$$

<u>e.g.</u> (1) $\Gamma_1 \supset \Gamma_2$, $\alpha = 1$.

$[\Gamma_1 \alpha \Gamma_2]_k : M_k(\Gamma_1) \to M_k(\Gamma_2)$

is simply inclusion.

(2) $\alpha^{-1}\Gamma_1 \alpha = \Gamma_2$

$$f[\Gamma_1 \alpha \Gamma_2]_k = f[\alpha]_k.$$

$$[\Gamma_1 \alpha \Gamma_2]_k : M_k(\Gamma_1) \to \underset{\substack{\| \\ M_k(\alpha^{-1}\Gamma_1\alpha)}}{M_k(\Gamma_2)}$$

is called <u>Hecke translation</u>.

(3) $\Gamma_1 \subset \Gamma_2$ $\alpha = 1$.

$$\Gamma_2 = \bigsqcup \Gamma_1 \beta_j$$

$[\Gamma_1 \alpha \Gamma_2]_k : M_k(\Gamma_1) \to M_k(\Gamma_2)$

is called a trace map.

<u>Recall:</u> Γ congruence subgp.

$Y_\Gamma = \Gamma \backslash \mathcal{H}$

$X_\Gamma = \Gamma \backslash \mathcal{H} \cup \{\text{cusps}\}$

$$\mathcal{S}_2(\Gamma) \simeq H^0(X_\Gamma, \Omega^1).$$

$$\begin{tikzcd}
\mathcal{M}_2(\Gamma_1) \arrow[r, "{[\Gamma_1]_k}"] \arrow[d, "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{M}_2(\Gamma_2) \arrow[d, "\wr"] \\
H^0(X_{\Gamma_1}, \Omega^1) \arrow[r] & H^0(X_{\Gamma_2}, \Omega^1)
\end{tikzcd}$$

trace map.

Hecke correspondence

$\Gamma_1, \Gamma_2, \ \alpha \in GL_2(\mathbb{Q})^+$

$X_1 = X_{\Gamma_1} = \Gamma_1 \backslash \mathcal{H} \cup \{\text{cusps}\}$

$X_2 = X_{\Gamma_2} = \Gamma_2 \backslash \mathcal{H} \cup \{\text{cusps}\}$

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ X_2 & & X_1 \end{tikzcd}$$ finite étale

<u>Lem:</u>

$$\Gamma_2 \longrightarrow \Gamma_1 \alpha \Gamma_2$$
$$\gamma \longmapsto \alpha\gamma$$

induces a bijection of cosets $\Gamma_1 \backslash \Gamma_1 \alpha \Gamma_2 \simeq {}_{\Gamma_3}\backslash \Gamma_2$

where $\Gamma_3 = \alpha^{-1} \Gamma_1 \alpha \cap \Gamma_2$.

$$\begin{tikzcd} \Gamma_3 \arrow[r, "\gamma \mapsto \alpha\gamma\alpha^{-1}"] \arrow[d] & \Gamma_3' := \Gamma_1 \cap \alpha \Gamma_2 \alpha^{-1} \arrow[d] \\ \Gamma_2 & \Gamma_1 \end{tikzcd}$$

induces.

$$\begin{tikzcd} X_3 \arrow[r] \arrow[d] \arrow[drr, dashed] & X_3' \arrow[dr] & \\ X_2 & & X_1 \end{tikzcd}$$

which is called a Hecke correspondence.

Goal: Study

$$\mathcal{M}_k(\Gamma_1(N))$$

$$\Gamma(N)$$

$$\cap|$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, \begin{matrix} c \equiv 0 \\ d \equiv 1 \\ (N) \end{matrix} \right\}.$$

$$\cap|$$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \, (N) \right\}$$

Construct Hecke operators that give endomorphisms of

$$\mathcal{M}_k(\Gamma_1(N)).$$

$$\Gamma_0(N) / \Gamma_1(N) \cong \left(\mathbb{Z}/N\mathbb{Z}\right)^\times$$

$$\Gamma_0(N) \longrightarrow (\mathbb{Z}/N\mathbb{Z})^\times$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto d \cdot (N)$$

$d \in (\mathbb{Z}/N\mathbb{Z})^\times$

$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.

$[\Gamma_1(N)\,\alpha\,\Gamma_1(N)]_k$ gives.

$\langle d \rangle : M_k(\Gamma_1(N)) \to M_k(\Gamma_1(N))$

is called a <u>diamond operator</u>.

$$(\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow GL(M_k(\Gamma_1(N)))$$

homomorphism

$$M_k(\Gamma_1(N)) = \bigoplus_{\chi} M_k(N, \chi)$$

$$\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$$

where $M_k(N, \chi)$ is the

eigenspace of $M_k(\Gamma_1(N))$ with eigenvalue χ.

In particular,

$$M_k(\Gamma_0(N)) = M_k(N, 1).$$

$\forall\, p$ prime,

$$T_p = \Gamma_1(N) \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}}_{=\alpha} \Gamma_1(N)$$

$$= \begin{cases} \displaystyle\bigcup_{j=0}^{p-1} \Gamma_1(N) \begin{bmatrix} 1 & j \\ 0 & p \end{bmatrix} & , \ p \mid N \\ \displaystyle\bigcup_{j=0}^{p-1} \Gamma_1(N) \begin{bmatrix} 1 & j \\ 0 & p \end{bmatrix} & , \ p \nmid N \\ \quad \cup\ \Gamma_1(N) \begin{bmatrix} m & n \\ N & p \end{bmatrix} \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} & \end{cases}$$

for $\forall \begin{bmatrix} m & n \\ N & p \end{bmatrix} \in SL_2(\mathbb{Z})$

Prop: Take $f \in M_k(\Gamma_1(N))$ with q-exp$_\infty$

$$f(\tau) = \sum_{n=0}^{\infty} a_n(f)\, q^n.$$

Then

$$(T_p f)(\tau) =$$

$$\sum_{n=0}^{\infty} a_{np}(f)\, q^n + 1_N(p)\, p^{k-1} \sum_{n=0}^{\infty} a_n(\langle p\rangle f)\, q^{np}$$

$$= \sum_{n=0}^{\infty} \left(a_{np}(f) + 1_N(p)\, p^{k-1} a_{\frac{n}{p}}(\langle p\rangle f) \right) q^n$$

$1_N : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ triv. $1_N(p)=0$ if $p \mid N$

$a_{\frac{n}{p}} = 0$ if $p \nmid n$.

__Pf:__ $(T_p f)(\tau)$ $\quad p \mid N$.

$$= \sum_{j=0}^{p-1} f\begin{bmatrix} 1 & j \\ 0 & p \end{bmatrix}_k (\tau)$$

$$= \sum_{j=0}^{p-1} p^{k-1}\, p^{-k} \cdot f\left(\frac{\tau+j}{p}\right).$$

$$= p^{-1} \sum_{j=0}^{p-1} \sum_{n=0}^{\infty} a_n(f)\, e^{2\pi i \frac{n(\tau+j)}{p}}.$$

$$= \sum_{n=0}^{\infty} a_{np}(f)\, q^n \qquad \square.$$

<u>Cor:</u> $\mathbb{C}\left[\langle d\rangle, T_p\right]$ $d \in (\mathbb{Z}/N\mathbb{Z})^*$, p prime.

$$\cap|$$

$$\mathrm{End}(\mathcal{M}_k(\Gamma_1(N))).$$

is commutative.

$\langle d\rangle,\quad d \in \mathbb{Z}.$

$$:\begin{cases} \langle d\rangle, & \text{if } (d,N)=1 \\ 0, & \text{if } (d,N)\neq 1. \end{cases}$$

$$T_{p^r} := T_p T_{p^{r-1}} - p^{k-1} \langle p \rangle T_{p^{r-2}}$$
$$(r \geq 2).$$

T_n is defined such that

$$T_{p_1^{r_1}} \cdots T_{p_s^{r_s}}$$

if $n = p_1^{r_1} \cdots p_s^{r_s}$.

$$\mathbb{C}[\langle d \rangle, T_n] \subseteq \mathrm{End}(M_k(\Gamma_1(N)))$$

$f, g \in M_k(\Gamma)$.

$$\langle f, g \rangle_\Gamma := \int_{\Gamma \backslash \mathcal{H}} f(\tau) \overline{g(\tau)} (\mathrm{Im}(\tau))^k \frac{dx\,dy}{y^2}$$

$$\mathrm{Im}(\tau) = y$$

which is abs. convergent

if one of f, g is a cusp form

$\langle \, , \rangle_\Gamma : S_k(\Gamma) \times S_k(\Gamma) \to \mathbb{C}$.

pos. def. Herm form

T : End$(S_k(\Gamma))$.

T^* adjoint of T.

$\langle Tf, g \rangle_\Gamma = \langle f, T^* g \rangle_\Gamma$.

<u>Prop</u>: Consider $S_k(\Gamma_1(N))$.
For $p \nmid N$, we have

(1) $\langle p \rangle^* = \langle p^{-1} \rangle$.

(2) $T_p^* = \langle p^{-1} \rangle \cdot T_p$.

$\Longrightarrow$ $\langle n \rangle$, T_n normal operator $(n, N) = 1$.

$$S_k(N, \chi) = \bigoplus_{\psi} S_k(N, \chi)[\psi].$$

ψ: character of the subalg of $\mathrm{End}(S_k(N, \chi))$ gen by T_n, $(n, N) = 1$