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K-moduli of curves on a quadric surfaces and K3 surfaces

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Motivation: birational geometry of moduli spaces.

A class of varieties (e.g. curves, K3 surfaces, hypersurf...)

Several well-known methods to construct compact moduli spaces.

1. Geometric invariant theory. (e.g. curves, hypersurfaces)
2. stable varieties/stable pairs (e.g. $\overline{M}_g$, KSB, A, etc...).
3. Baily-Borel compactifications (e.g. K3, abelian var. ...)

K-stability : stability theory for (log) Fano's.

<u>Slogan</u>. K-stability moduli spaces of log Fano pairs interpolate between GIT and KSB/Baily-Borel via wall crossing by changing coefficients.

[Ascher-DeVleming-L. '19], $(\mathbb{P}^2, c\cdot C_d)$ $d = $ degree ≥ 4.

$\rightsquigarrow$ K-moduli space $\overline{P}^K_{d,c}$, $c \in (0, \frac{3}{d})$.

$\nearrow$ c small, $\cong \overline{P}^{GIT}_d$

$\searrow$ c close to $\frac{3}{d}$, close to Hacking moduli space (KSB).

e.g. $d = 4, 6$, then $\overline{P}^K_{d, \frac{3}{d}-\varepsilon} \to P^*_d$.

<u>Today</u> : curves on quadric surfaces, in particular

$(2,4)$ - c.i. $\subseteq \mathbb{P}^3$.

<u>Set-up</u>. $Q \subseteq \mathbb{P}^3$ quadric surface, normal. $(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}(1,1,2))$.

$C \in |\mathcal{O}_Q(4)|$. $(2,4)$ - complete intersection.

Assume C has ADE sing.

Then $\pi : S \xrightarrow{2:1} (Q, \frac{1}{2}C)$ produces a K3 surface S, (with ADE sing).

with polarization $L = \pi^* \mathcal{O}_Q(1)$.

$\Rightarrow (S, L)$ is a pol. hyp. K3 of deg 4. (Indeed, all such K3 arise in this way).

$\rightsquigarrow$ coarse moduli space $\quad M = \{(S,L)\}/\cong \; = \{(Q, \tfrac{1}{2}C)\}/\cong$

normal $\quad$ (2,4)-c.i. ADE sg.

$\dim M = 18.$

① GIT. $\quad$ A smooth $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, $\quad C \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(4,4)|.$

$$\overline{M}^{GIT} := |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(4,4)| /\!/ \mathrm{Aut}(\mathbb{P}^1\times\mathbb{P}^1). \quad \text{(good moduli).}$$

$$M^0 := \{(Q,C)\in M \mid Q \text{ smooth}\}/\cong$$

$$M^0 \cong \{(\mathbb{P}^1\times\mathbb{P}^1, C) \mid C \text{ has ADE sg}\}. \xhookrightarrow{\text{open}} \overline{M}^{GIT}$$

$$\operatorname{codim}_{\overline{M}^{GIT}} \overline{M}^{GIT} \setminus M^0 \geq 2.$$

$$H = M \setminus M^0 = \{(Q,C) \in M \mid Q \cong \mathbb{P}(1,1,2)\}. \text{ divisor in } M.$$

② Baily-Borel compactification.

(S, L) : polarized (ADE) hyp. K3 of deg 4.

lattice polarized.

$$\rightsquigarrow \quad \mathcal{D}/\Gamma \xleftarrow[\text{period map}]{\simeq} M.$$

BB comp. $\quad M^* = (\mathcal{D}/\Gamma)^* := \operatorname{Proj} R(\mathcal{D}/\Gamma, \lambda)$

$\dim M^* \setminus M \leq 1.$

- K-moduli spaces.

$(\mathbb{P}^1\times\mathbb{P}^1, c\cdot C)$ log Fano. $C\in|\mathcal{O}(4,4)|$, $c\in(0,\frac{1}{2})\cap\mathbb{Q}$.

If C has only ADE sing., then interpolation $\Rightarrow$

$(\mathbb{P}^1\times\mathbb{P}^1, c\cdot C)$ is K-stable $\forall\, c\in(0,\frac{1}{2})$.

<u>Def</u> [ADL'19]. Let $\overline{M}^K_c$ be the proper (→ proj (Xu-Zhuang)) moduli space of K-polystable pairs (X, cD) admitting $\mathbb{Q}$-Gor smoothly to $(\mathbb{P}^1\times\mathbb{P}^1, c\cdot C_t)$.

$\exists$ open immersion $M^0 \hookrightarrow \overline{M}^K_c$ $\forall\, c\in(0,\frac{1}{2})$.

Thm 1 (ADL) For $\forall c \in (0, \frac{1}{8})$, $\overline{M}^K_c \cong \overline{M}^{GLT}$.

Thm 2 (ADL) $\exists$ explicit walls $0 < c_1 < \cdots < c_k < \frac{1}{2}$.

s.t. (1) $\overline{M}^K_c$ is indep. of $c \in (c_i, c_{i+1})$.

étale locally in general

(2)
$$\begin{tikzcd} \overline{M}^K_{c_i-\varepsilon} \arrow[rr, dashed, leftrightarrow, "bir"] \arrow[dr] & & \overline{M}^K_{c_i+\varepsilon} \arrow[dl] \\ & \overline{M}^K_{c_i} & \end{tikzcd}$$
globally VGIT diagram.

(3) $\exists\ \overline{M}^K_{\frac{1}{2}-\varepsilon} \xrightarrow{bir} M^*$ $\mathbb{Q}$-Cartierization of $H^* \subseteq M^*$.

(Looijenga semitoric modif.).

<u>Thm 3</u> (ADL) $\forall c \in (0, \frac{1}{2})$, only surface appearing in $\overline{M}^K_c$ is either $\mathbb{P}^1 \times \mathbb{P}^1$, or $\mathbb{P}(1,1,2)$.

$\rightsquigarrow$ enable us to compare K-moduli with VGIT (Laza-O'Grady).

<u>Remark</u>. thm 3 is false for other degrees.

E.g. (3,3) curves on $\mathbb{P}^1 \times \mathbb{P}^1$. $\rightsquigarrow$ when $c = \frac{1}{2}$, $\mathbb{P}(1,2,9)$ will occur in K-moduli (Odaka-Spotti-Sun '16 in del Pezzo of deg 1).

§ Normalized volume estimates.

<u>Def</u>. $x \in (X, D)$ klt sing, $n = \dim_{\mathbb{C}} X$.

$$\begin{tikzcd} E \arrow[r, hook, "\text{prime div}"] \arrow[d] & Y \arrow[d, "f"', "\text{bir}"] \\ x \in & X \end{tikzcd}$$

Log discrepancy: $A_{X,D}(E) := 1 + \mathrm{coeff}_E(K_Y - f^*(K_X + D))$.

Volume: $\mathrm{vol}(E) := \lim\limits_{m \to \infty} \dfrac{\dim_{\mathbb{C}} \mathcal{O}_{X,x} / \{g \mid \mathrm{ord}_E(g) \geq m\}}{m^n / n!}$

C. Li's normalized volume:

$$\widehat{\mathrm{vol}}(E) := A_{X,D}(E)^n \cdot \mathrm{vol}(E). \quad \widehat{\mathrm{vol}}(x, X, D) = \inf_{E / x \in (X, D)} \widehat{\mathrm{vol}}(E)$$

Thm 4 (L'16) Fujita'15 Suppose (X, D) log Fano, K-semistable.

Then $\forall x \in X$ closed pt,

$$(-K_X - D)^n \leq \left(1 + \frac{1}{n}\right)^n \cdot \widehat{\mathrm{vol}}(x, X, D). \quad (*)$$

Ex. For quotient sing $x \in X \cong^{an} (0 \in \mathbb{A}^n / G)$

then $\widehat{\mathrm{vol}}(x, \cancel{X}, 0) = \dfrac{n^n}{|G|}$.

$(*)$ becomes $(-K_X - D)^n \leq \dfrac{(n+1)^n}{|G|}.$ vol of $\mathbb{P}^n$.

<u>Proof of thm 1</u> ($\forall c \in (0, \frac{1}{8})$, $\overline{M}^K_c \cong \overline{M}^{GIT}$).

It suffices to show only $\mathbb{P}^1 \times \mathbb{P}^1$ appears in K-moduli.

(Paul-Tian $\Rightarrow$ $\overline{M}^K_c \overset{bir}{\longleftrightarrow} \overline{M}^{GIT}$ $\Rightarrow$ $\cong$).

Apply thm 4, if $(X, cD) \in \overline{M}^K_c$ is K-ss, then

$$\frac{9}{2} < (-K_X - cD)^2 = (-K_{\mathbb{P}^1 \times \mathbb{P}^1} - c \cdot \mathcal{O}(4,4))^2 = 2(2-4c)^2$$

$$\underset{\text{thm 4}}{\leq} \frac{9}{|G|} \qquad G = \text{orbifold group at } x \in X.$$

$\Rightarrow \forall x \in X$ is smooth, hence X is smooth, so $X \cong \mathbb{P}^1 \times \mathbb{P}^1$.

#.

Proof of thm 3 (Only $\mathbb{P}^1\times\mathbb{P}^1$ & $\mathbb{P}(1,1,2)$ can appear in K-mod)

Case 1 $\quad c < \dfrac{4-\sqrt{2}}{8}$. $\quad (-K_X - cD)^2 > 1$.

thm 4 $\Rightarrow (-K_X - cD)^2 \leq \dfrac{9}{|G|} \Rightarrow |G| < 9$.

Use classification of degeneration of $\mathbb{P}^1\times\mathbb{P}^1$ (Hacking-Prokhorov)

Gor index of $X \leq 2$. $\rightsquigarrow$ (Alexeev-Nikulin, Nakayama)

Case 2 $\quad c \geq \dfrac{4-\sqrt{2}}{8}$ $\quad (-K_X - cD)^2$ can be very small

Consider $\widehat{\mathrm{vol}}(x, X, cD)$ for $x \in \mathrm{Supp}(D)$. $\Rightarrow$ ✓ (only works for (4,4)).

If $x \notin \mathrm{Supp}(D)$, $D \sim -2K_X$ Cartier at $x \Rightarrow$ Gor index at $x \leq 2 \Rightarrow$ ✓ #.

$\underline{\text{Ex}}.$ $c_1 = \frac{1}{8}$ Replace $(\mathbb{P}^1 \times \mathbb{P}^1, 4A)$ A smooth $(1,1)$-curve.

conic curve.

$c = \frac{1}{8} - \varepsilon$ K-ps

$(\overset{x\ y\ z}{\mathbb{P}(1,1,2)}, D) \overset{c = \frac{1}{8}+\varepsilon}{\rightsquigarrow} (\mathbb{P}(1,1,2), 4A_0)$

$D = (z^4 = f_8(x,y))$ f_8 GIT polystable.

$\underline{\text{Thm 5}}$ [ADL]. $\overline{M}^K_c \cong \overline{M}(t)$, $t = \frac{3c}{2c+2}$.

$c \in (0, \frac{1}{2})$ $t \in (0, \frac{1}{2}]$.

L-O'G VGIT of slope t