Representations of Reductive Groups

will define later

I. Structure theory

Let G be a connected alg gp.

"Defn" G is a unipotent group if

$\exists$ sequence of normal subgps

$1 = G_0 \subset G_1 \subset \cdots \subset G_k = G$

s.t. $G_i / G_{i-1} \simeq \mathbb{G}_a$

G is solvable if

- $\exists$ torus $T \subset G$
- $\exists$ normal unipotent subgp $U \subset G$

such $G = T \ltimes U$

WARNING: These are not the usual defns - usually stated as thms.

Examples.

1) Let $U = \left\{ \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{bmatrix} \right\} \subset GL_n$

- unipotent

2) Let $B = \left\{ \begin{bmatrix} * & & 0 \\ & \ddots & \\ * & & * \end{bmatrix} \right\} \subset GL_n$

- solvable - $B = T \ltimes U$

where $T = \begin{bmatrix} * & & \\ & \ddots & \\ & & * \end{bmatrix}$.

3) Any unipotent gp is solvable (torus part is trivial)

4) Any torus is solvable (unipotent part is trivial)

5) (Nontrivial thm) Any connected commutative alg gp is solvable.

Defn. Let G be a conn alg gp.

The radical of G, denoted $R(G)$ is the (unique) maximal normal solvable subgp of G.

The unipotent radical $R_u(G)$ is the max'l normal unip subgp of G.

G is semisimple if $R(G) = \text{triv}$.

reductive if $R_u(G) = \text{triv}$.

$R_u(G) \subset R(G)$

semisimple $\Rightarrow$ reductive.

Lemma. For any G, $R_u(G)$ acts trivially in any irreducible rep of G.

Reductive gps are the "nicest" groups to study.

Examples

1) Any torus is reductive. (not semisimple)

2) GL_n is reductive, but not semisimple

$R(GL_n)$ = its center $= \left\{ \begin{bmatrix} a & & \\ & a & \\ & & \ddots \\ & & & a \end{bmatrix} \middle| a \in \mathbb{k}^\times \right\}$.

3) SL_n is semisimple

4) Let V be a vec sp equipped with a nondegenerate bilinear form

$\langle -,- \rangle : V \times V \longrightarrow \mathbb{k}$

Let

$G = \{ g \in GL(V) \mid \langle gv, gw \rangle = \langle v, w \rangle \ \forall v, w \in V \}$.

If $\langle , \rangle$ is symmetric, then

$G = O(V)$ = orthogonal gp

$G \cap SL(V) = SO(V)$ = special orthogonal gp

If $\langle , \rangle$ is skew-symmetric then

$G = Sp(V)$ = symplectic group

$SO(V)$ & $Sp(V)$ are semisimple.

II. Structure theory for reductive groups

Inside any conn. alg gp G, can find a maximal torus, i.e. a torus subgp, not contained in any larger torus subgp

Fact. All maximal tori in G are conjugate (so all have same dim)

Idea - Reps of a torus are "well-understood"

- Try to understand reps of G by restricting to a max torus

T and G are a bit "too far apart"

Intermediate role: played by

Defn A Borel subgp of G is a max'l connected solvable subgp.

Facts 0) Any max torus is contained in a Borel subgp.

1) All Borel subgps are conjugate

2) Normalizers: $N_G(\underset{\text{Borel}}{B}) = B$

$[N_G(\underset{\text{max torus}}{T}) : T]$ is finite.

When studying reductive gps:
usually: Fix $G \supset B \supset T$
$\uparrow$ Borel $\quad$ $\uparrow$ max torus

Defn The Weyl group of G (with respect to T) is the finite gp $W = N_G(T)/T$

$N_G(T)$ acts on T by conjugation
$\leadsto$ W acts on T by "conjugation"
Also: W acts on $X(T)$ by
For $w \in W$, $\lambda \in X(T)$

$$(w \cdot \lambda)(\underset{\substack{\Uparrow \\ T}}{t}) = \lambda(\underset{\substack{\uparrow \\ \text{conjugation action.}}}{w^{-1} \cdot t})$$

Example For GL_n, take $\overset{\text{can}}{\leftarrow} X(T) = \mathbb{Z}^n$

$$B = \begin{bmatrix} * & & 0 \\ & \ddots & \\ * & & * \end{bmatrix} \supset T = \begin{bmatrix} * & & \\ & \ddots & \\ & & * \end{bmatrix}$$

$N_G(T)$ = matrices with exactly 1 nonzero entry in each row & each column.

$n=2$: $N_G(T) = \left\{\begin{bmatrix} * & \\ & * \end{bmatrix}\right\} \cup \left\{\begin{bmatrix} & * \\ * & \end{bmatrix}\right\}$

For GL_n,

$W = N_G(T)/T \simeq S_n$ = symmetric gp on n letters.

$S_n \circlearrowright \mathbb{Z}^n$ by permuting coordinates.

III. Weights and roots

The Lie algebra of G is the tangent space to G at the identity elt $1 \in G$.
Notation: $\mathfrak{g}$ - vec sp.

G acts on itself by conjugation
Differentiate $\leadsto$
G-rep on $\mathfrak{g}$, called the adjoint representation

Decompose $\mathfrak{g}$ into weight spaces as a T-rep.

$$\mathfrak{g} = \bigoplus_{\alpha \in X(T)} \mathfrak{g}_\alpha = \underset{\substack{\| \\ \text{Lie alg.} \\ \text{of } T.}}{\mathfrak{g}_0} \oplus \bigoplus_{\substack{\alpha \in X(T) \\ \alpha \neq 0}} \mathfrak{g}_\alpha$$

The nonzero α such that $\mathfrak{g}_\alpha \neq 0$ are called roots.

Set of roots: $\Phi = \underset{\text{positive}}{\Phi^+} \cup \underset{\text{negative}}{\Phi^-}$

$\Phi^- = \{\alpha \in \Phi \mid \mathfrak{g}_\alpha \subset \text{Lie alg. of } B\}$.

$\Phi^+ = \{\alpha \in \Phi \mid \mathfrak{g}_\alpha \not\subset$ " " $\}$.

Example $GL_n \supset B \supset T$ $\quad W = S_n$
$\begin{bmatrix} * & * \\ & * \end{bmatrix}$ $\begin{bmatrix} * & & \\ & \ddots & \\ & & * \end{bmatrix}$

$X(T) = \mathbb{Z}^n$

Let $\varepsilon_i = (0, \cdots, 0, 1, 0, \cdots, 0) \in X(T)$
i^{th} coordinate.

Exercise
$\Phi = \{ \varepsilon_i - \varepsilon_j \mid 1 \leq i, j \leq n, \ i \neq j \}$
$\Phi^+ = \{ \varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n \}$.

Example $SL_2 \supset B \supset T$ $\quad W = S_2$
$\begin{bmatrix} a & \\ c & a^{-1} \end{bmatrix}$ $\begin{bmatrix} a & \\ & a^{-1} \end{bmatrix}$

$X(T) = \mathbb{Z}$

Roots: $\Phi = \{\pm 2\}$ $\quad \Phi^+ = \{2\}$.

Equip $X(T)$ with a partial order $\preceq$:

$\lambda \preceq \mu$ if

$\mu - \lambda \in \mathbb{Z}_{\geq 0} \Phi^+$ = nonneg. integer linear comb. of pos. roots

Definition. A character $\lambda \in X(T)$ is called **dominant** if $\forall w \in W$, $w\lambda \preceq \lambda$.

Set of dominant wts: X^+.

Examples.
GL_n : $X^+ = \{(a_1, \cdots, a_n) \in \mathbb{Z}^n \mid a_1 \geq \cdots \geq a_n\}$.

SL_2 : $X^+ = \mathbb{Z}_{\geq 0}$

IV. Representations

Let G be a reductive gp $G \supset B \supset T$, W.

Prop $\operatorname{ind}_B^G \Bbbk_\lambda = 0$ unless $\lambda \in X^+$.

(Here $\Bbbk_\lambda$ is the 1-dim'l B-rep on which T acts by λ.)

For $\lambda \in X^+$, let $V_\lambda = \operatorname{ind}_B^G \Bbbk_\lambda$.

Prop. $\dim V_\lambda < \infty$. It has a unique irreducible subrep.
Call that rep L_λ.

Thm ① Every irred rep of G is isomorphic to L_λ for some $\lambda \in X^+$
② If $\lambda \neq \mu$, $L_\lambda \not\cong L_\mu$.

Thm. If $\Bbbk = \mathbb{C}$, $L_\lambda = V_\lambda$.

Character formula for V_λ.

Let
$$\rho = \underbrace{\tfrac{1}{2} \underbrace{\sum_{\alpha \in \Phi^+} \alpha}_{\in X(T)}}_{\in \mathbb{Q} \otimes_{\mathbb{Z}} X(T).}$$

<u>Thm</u> (Weyl's character formula)
As a T-rep,
$$\underset{\substack{\uparrow \\ \operatorname{ind}_B^G \Bbbk_\lambda}}{\operatorname{ch} V_\lambda} = \underbrace{\frac{\sum_{w \in W} \operatorname{sgn}(w) e^{w(\lambda+\rho)}}{\sum_{w \in W} \operatorname{sgn}(w) e^{w\rho}}}_{\substack{\text{a well-defined element} \\ \text{of } \mathbb{Z}[X(T)].}}$$

<u>Exercise</u> Confirm for SL_2.