Lecture II.

Given a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$. We consider $E_{\mathcal{F}}$ $\cancel{M_{\mathcal{F}}} = \bigoplus_j H^0(\mathcal{F}(j))$. Since $M_{\mathcal{F}}$ determines $\mathcal{F}$. It is natural to ask how can one control the complexity of $\mathcal{F}$ or $M_{\mathcal{F}}$. What can one say about the shape of the minimal resolution of $\cancel{M_{\mathcal{F}}}$ $E_{\mathcal{F}}$. An important tool in this area is the Castelnuovo-Mumford regularity introduced by Mumford. It is interesting to compare the regularity of $\mathcal{I}_X$ with invariants such as degree of X.

or the degrees of the defining equations.

Let V be a $(n+1)$-dimension vector space over k. Set $\mathbb{P}(V) = \mathbb{P}^n = \mathbb{P}$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}$.

Definition 2.1. Given an integer, one says that $\mathcal{F}$ is m-regular, if $H^i(\mathcal{F}(m-i)) = H^i(\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}}(m-i)) = 0$ for $i > 0$.

The following is the basic theorem.

Theorem 2.2. (Mumford) Assume $\mathcal{F}$ is m-regular. Then for every positive integer l.

(i) $\mathcal{F}(m)$ is globally generated

(ii) The multiplication map;

$$H^0(\mathcal{F}(m)) \otimes H^0(\mathcal{O}_{\mathbb{P}}(l)) \longrightarrow H^0(\mathcal{F}(m+l))$$

is surjective.

(iii) $\mathcal{F}$ is $(m+l)$-regular. This means $H^i(\mathcal{F}(j)) = 0$ for $i > 0$ and $j \geq m-i$.

Proof Set ~~$M = \Omega_{\mathbb{P}^n}(1)$~~. There is the Euler sequence ~~$0 \to M \to V \otimes \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_{\mathbb{P}}(1) \to$~~

$$0 \to \Omega_{\mathbb{P}} \to V \otimes \mathcal{O}_{\mathbb{P}}(-1) \to \mathcal{O} \to 0$$

There is the Koszul complex

$$\cancel{0 \to \bigwedge^{n+1} V \otimes \mathcal{O}(-n) \to \cdots \to V \otimes \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_{\mathbb{P}}(1) \to 0.}$$

$$0 \to \bigwedge^{n+1} V \otimes \mathcal{O}_{\mathbb{P}}(-n-1) \to \cdots \bigwedge^1 V \otimes \mathcal{O}(-1) \to \to \mathcal{O}_{\mathbb{P}} \to 0$$

Proof. Let's prove (ii). Tensor the Koszul complex by $\mathcal{I}(m+1)$.
Consider the case $l=1$.

$$0 \to \Omega_{\mathbb{P}} \otimes \mathcal{I}(m+1) \to V \otimes \mathcal{I}(m) \to \mathcal{I}(m+1) \to 0$$

So enough to show $H^1(\Omega_{\mathbb{P}} \otimes \mathcal{I}(m+1)) = 0$

From

$$0 \to \wedge^2 \Omega_{\mathbb{P}} \otimes \mathcal{I}(m+1) \to \wedge^2 V \otimes \mathcal{I}(m-1) \to \Omega_{\mathbb{P}} \otimes \mathcal{I}(m+1) \to 0$$

Since $H^1(\mathcal{I}(m-1)) = 0$, we see $H^1(\Omega_{\mathbb{P}} \otimes \mathcal{I}(m+1))$

$\hookrightarrow H^2(\Omega^2_{\mathbb{P}} \otimes \mathcal{I}(m+1))$

Repeat the same argument, we see

$H^1(\Omega_{\mathbb{P}} \otimes \mathcal{I}(m+1)) \hookrightarrow H^n(\Omega^n_{\mathbb{P}} \otimes \mathcal{I}(m+1))$.

But $\Omega^n_{\mathbb{P}} \simeq \mathcal{O}_{\mathbb{P}}(-n-1)$, So $H^n(\Omega^n \otimes \mathcal{I}(m+1))$

$\simeq H^n(\mathcal{I}(m-n)) = 0$. So (ii) is true for $l=1$.

Next we show $l=1$ for (iii), Again by chasing cohomology $H^i(\mathcal{I}(m+1-i)) \hookrightarrow H^{i+1}(\Omega^1 \otimes \mathcal{I}(m+1-i)) \cdots$

$\hookrightarrow H^n(\Omega^{n-i} \otimes \mathcal{I}(m+1-i))$. The last group is the quotient $H^n(\wedge^{n-i+1} V \otimes \mathcal{O}_{\mathbb{P}}(-n+i-1) \otimes \mathcal{I}(m+1-i))$

$\simeq H^n(\mathcal{I}(m-n)) = 0$. This proves iii

By induction, we prove (ii) and (iii) for all $l \geq 1$.

For (i). We choose $l \gg 0$, such that $\mathcal{F}(m+l)$ is globally generated. This is possible by Serre's theorem. Consider the diagram:

$$\begin{array}{ccc}
\left(H^0(\mathcal{F}(m)) \otimes H^0(\mathcal{O}_{\mathbb{P}}(l))\right) \otimes \mathcal{O}_{\mathbb{P}} & \longrightarrow & H^0(\mathcal{F}(m)) \otimes \cancel{H^0(\mathcal{O}_{\mathbb{P}}(l))} \otimes \mathcal{O}_{\mathbb{P}}(l) \\
\varphi \downarrow & & \downarrow \tau \\
H^0(\mathcal{F}(m+l)) \otimes \mathcal{O}_{\mathbb{P}} & \xrightarrow{\psi} & \mathcal{F}(m+l)
\end{array}$$

φ is surjective by (ii) and ψ is surjective by our choice of l. τ is equivalent $\left(H^0(\mathcal{F}(m)) \otimes \mathcal{O}_{\mathbb{P}} \longrightarrow \mathcal{F}(m)\right) \otimes \mathcal{O}_{\mathbb{P}}(l)$.

We conclude τ is surjective. Hence $\mathcal{F}(m)$ is globally generated.

Remark 2.3. 2.2 (ii) is equivalent to the following: All the minimal generators of $E_{\mathcal{Y}}$ are of degree less than or equal m.

Exercise 2.4. Let i_0 be ~~an~~ a non-negative integer. ~~great than or equal to one~~

We say $\mathcal{Y}$ is m-regular in degree $> i_0$ if $H^i(\mathcal{Y}(m-i)) = 0$ for $i > i_0$.

(i) Show that $\mathcal{Y}$ is ~~reg~~ $(m+l)$-regular ~~for~~ in degree $> i_0$ for all $l \geq 0$.

(ii) ~~The mul~~ ~~$H^0(\mathcal{O}_{\mathbb{P}}(1)) \otimes H^{i_0}(\mathcal{Y})$.~~

The multiplication map

$$H^0(\mathcal{O}_{\mathbb{P}}(1)) \otimes H^{i_0}(\mathcal{Y}(m)) \longrightarrow H^{i_0}(\mathcal{Y}(m+1))$$

is surjective.

Excercise 2.5.

Suppose X is a projective variety of dimension n. Suppose $\mathcal{L}$ is an globally generated line bundle on X. A coherent sheaf $\mathcal{F}$ on X is said to be m-regular with respect to $\mathcal{L}$, if $H^i(\mathcal{F} \otimes \mathcal{L}^{\otimes m-i}) = 0$ for all $i > 0$. ~~St~~

(i) Show that $\mathcal{L}$ is $(m+l)$-regular for $l \geq 0$.

(ii) $H^0(\mathcal{L}) \otimes H^0(\mathcal{F}(m)) \longrightarrow H^0(\mathcal{F}(m+1))$ is surjective

(iii) $\mathcal{F}(m)$ is globally generated.

Excercise 2.6.

Show that (iii) of Excercise 2.5, would fail, if we only assume $\mathcal{L}$ is globally generated but not ample.

Excercise 2.~~8~~7. Let $X = \mathbb{P} \in \mathbb{P}^n$ be a point

~~$\bigoplus H^0(\mathcal{O}_{\mathbb{P}}(\ast) \otimes \mathcal{O}_{\mathbb{P}})$~~ $H^0(\mathcal{O}_X(m)) = k$

for all m. So $\bigoplus_{m \in \mathbb{Z}} H^0(\mathcal{O}_X(m))$ is not a finitely generated S module.

This is why we set $E_{\mathcal{G}} = \bigoplus_{l >> -\infty} H^0(\mathcal{G}(l))$.

Then $E_{\mathcal{G}}$ is finitely generated.

Theorem 2.8 (Regularity and syzygies). Let $\mathcal{F}$ be a m-regular coherent sheaf on $\mathbb{P}^n$. Consider the minimal graded free resolution of $E_{\mathcal{F}}$.

$$0 \to F_{n+1} \quad \cdots\cdots \; F_1 \to \cdots F_0 \to E_{\mathcal{F}} \to 0$$

We F_i as $\bigoplus_j S(-j)^{\beta_{ij}}$.

Then $\beta_{ij} = 0$ if $j > m+i$.

Proof

Consider the corresp ~~sequen~~ complex on of coherent sheaves on $\mathbb{P}^n$

$$0 \to \mathcal{F}_1 \to \bigoplus_j \mathcal{O}(-j)^{\beta_{0j}} \xrightarrow{\varphi} \mathcal{F} \to 0.$$

By construct. H^1 ~~$(\mathcal{F}_1(m))$~~ H^0 ~~$(\varphi) \otimes \mathcal{O}_{\mathbb{P}}$~~

$H^1(\mathcal{F}(m)) = 0$. Since $H^0(\varphi)$ is essentl surject.

For $j \geq 1$, $H^i(\mathcal{F}_1(m+1-i)) \simeq H^{i-1}(\mathcal{F}(m-i))$ $= 0$. So $\mathcal{F}_1$ ~~is~~ is $m+1$ - regular

We've

$$0 \to \oplus E_{\mathcal{F}_1} \to \oplus S(-j)^{\beta_{0j}} \to E_{\mathcal{F}} \to 0$$

Now we repeat the argument to see the shape of the minimal resolution.

Remark 2.9. The theorem says that all the minimal syz. generators of $E_{\mathcal{F}}$ have degree $\leq m$. All the minimal first syzygies have degree $\leq m+1$ etc.

Corollary 2.10 Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$. Then $\mathcal{F}$ is m-regular if and only if there is an exact Complex of the following form.

$$\cdots \oplus \mathcal{O}_{\mathbb{P}}(-m-2) \xrightarrow{\varphi_2} \oplus \mathcal{O}_{\mathbb{P}}(-m-1) \xrightarrow{\varphi_1} \oplus \mathcal{O}_{\mathbb{P}}(-m) \xrightarrow{\varphi_0} \mathcal{F} \to 0$$

Proof $\Rightarrow$ Assume $\mathcal{F}$ is m-regular. Then $\mathcal{F}(m)$ is globally generated. Then consider

$$0 \to \mathcal{F}_1 \to H^0(\mathcal{F}(m)) \otimes \mathcal{O}_{\mathbb{P}}(-m) \longrightarrow \mathcal{F} \to 0.$$

~~$\mathcal{F}_1$ is $(m+1)$-regular.~~

One Checks $H^1(\mathcal{F}(m)) = 0$ and $\mathcal{F}_1$ is $(m+1)$-regular.

Now just repeat the construct to construct the exact Complex

$\Leftarrow$ Assume the existence of the complex

Set $\mathcal{G}_i = \ker \varphi_{i-1}$, for $i = 1, 2, \ldots$

~~Then $H^i(\mathcal{F}(m-i)) \Longleftrightarrow H^{i-1}(\mathcal{G}_1)$~~

Starting from the complex

$\Leftarrow$ Set $\mathcal{G}_i = \operatorname{Ker} \varphi_{i-1}$.

Then $H^1(\cancel{\mathcal{F}(m-1)}) \simeq \cancel{H^1(\mathcal{G}_0}(m-$

Then $H^i(\mathcal{F}(m-i)) \hookrightarrow H^{i+1}(\mathcal{G}_1(m-i))$.

$$\hookrightarrow \cancel{H^n(\mathcal{G}}\, H^{i+(n-i)}(\mathcal{G}_{n-i}(m-i))$$

The last is the quotn

$$\oplus H^n(\mathcal{O}_{\mathbb{P}}(\cancel{-m}\, -n+i, -m' + m-i)) = \oplus H^n(\mathcal{O}_{\mathbb{P}}(-n)) = 0.$$

Remark 2.11. The above complex may be infinitely long.

Starting from the complex

$\Leftarrow$ Set $\mathcal{G}_1 = \operatorname{Ker} \varphi_{1-1}$.

Then ~~$H^1(\mathcal{F}(m-1)) \simeq H^1(\mathcal{G}_0(m-$~~

Then $H^i(\mathcal{F}(m-i)) \hookrightarrow H^{i+1}(\mathcal{G}_1(m-i))$.

$\hookrightarrow \cancel{H^n(\mathcal{G}} \, H^{i+(n-i)}(\mathcal{G}_{n-i}(m-i))$

The last group is a ~~the~~ quotient of

$\bigoplus H^n(\mathcal{O}_{\mathbb{P}}(-n+i-m'+m-i)) = \bigoplus H^n(\mathcal{O}_{\mathbb{P}}(-n)) = 0$.

Remark 2.11. The above complex may be infinitely long.

Definition 2.12. Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$. $\mathcal{F}$ admits a m-linear resolution if there is an exact complex of the form

$$\cdots \to \bigoplus \mathcal{O}_{\mathbb{P}}(-m-2) \xrightarrow{\varphi_2} \bigoplus \mathcal{O}_{\mathbb{P}}(-m-1) \xrightarrow{\varphi_1} \bigoplus \mathcal{O}_{\mathbb{P}}(-m) \to \mathcal{F} \to 0.$$

Observe that each φ_i is a matrix of linear forms.

Theorem 2.11. Say, $\mathcal{F}$ admits a m-linear resolution if and only if $\mathcal{F}$ is m-regular.

Theorem 2.13. (Regularity of tensor product).
Let $\mathcal{E}$ be locally free sheaf on $\mathbb{P}^n$ ~~and~~ and $\mathcal{F}$ is an arbitrary coherent sheaf on $\mathbb{P}^n$. ~~Then~~ Assume $\mathcal{E}$ is m_1-regular and $\mathcal{F}$ is m_2-regular. Then $\mathcal{E} \otimes \mathcal{F}$ is (m_1+m_2)-regular. Moreover the natural mapping

$$H^0(\mathcal{E}(m_1)) \otimes H^0(\mathcal{F}(m_2)) \longrightarrow H^0(\mathcal{E} \otimes \mathcal{F}(m_1+m_2))$$

is surjective.

Proof. Consider a m_2-linear resolution of $\mathcal{F}$ of the form:

$$\cdots\cdots \longrightarrow \bigoplus \mathcal{O}_{\mathbb{P}}(-1) \longrightarrow H^0(\mathcal{F}(m_2)) \cdot \mathcal{O}_{\mathbb{P}} \longrightarrow \mathcal{F}(m_2)$$

Tensor the above complex by $\mathcal{E}(m_1)$.
Compute cohomology would yield the result.

Exercise 2.14. Check the details of the above ~~Pro~~ proof.

Corollary 2.15 Assume $\mathcal{E}$ is a m-regular locally free sheaf on $\mathbb{P}^n$. Then

(i) $T^p(\mathcal{E})$, the p-fold tensor product of $\mathcal{E}$ is mp-regular

(ii) Assume char $k = 0$. Then $S^p(\mathcal{E})$ and $\Lambda^p\mathcal{E}$ are also mp-regular.

Proof. (i) follows from Theorem 2.13.

(ii) In characteristic zero, $\Lambda^p\mathcal{E}$ and $S^p\mathcal{E}$ are direct summands of $T^p(\mathcal{E})$.

Example 2.16. (Regularity of
Assume that there is a finite. Assume that
a complex of the form

$$W_\bullet \quad \cdots \quad W_2 \otimes \mathcal{O}_{\mathbb{P}}(-m-2) \xrightarrow{\varphi_2} W_1 \otimes \mathcal{O}_{\mathbb{P}}(-m-1) \xrightarrow{\varphi_1} W_0 \otimes \mathcal{O}_{\mathbb{P}}(-m) \xrightarrow{\varepsilon} \mathcal{F}$$

Assume that (i) ε is surjective

(ii) The homology of the complex have zero dimensional support.

Then $\mathcal{F}$ is m-regular.

Excercise 2.17. Check the details for Example 2.16.

Exercise. ~~Example~~ 2.18. Assume Z is a zero dimension closed subscheme in $\mathbb{P}^n$ with ideal sheaf $\mathcal{I}_Z$, where $\mathcal{I}_Z$ is a-regular. Suppose $\mathcal{F}$ is a m-regular coherent sheaf on $\mathbb{P}^n$. Then the natural map is

$$H^0(\mathcal{F}(m+a-1)) \longrightarrow H^0(\mathcal{F}(m+a-1)\otimes \mathcal{O}_Z \to$$

(Hint. There is an exact sequence

$$0 \to \mathrm{Tor}_1(\mathcal{F}, \mathcal{O}_Z) \to \mathcal{F}\otimes \mathcal{I}_Z \to \mathcal{F}\cdot \mathcal{I}_Z \to 0.$$

where $\mathrm{Tor}_1(\mathcal{F}, \mathcal{O}_Z)$ has zero dimension support).

Proposition 2.19. Let $Z \subseteq \mathbb{P}^n$ be a length d closed subscheme ~~of~~ Let $\mathcal{I}_Z$ be ~~to~~ the ideal sheaf Z. Then $\mathcal{I}_Z$ is always d-regul. $\mathcal{I}_Z$ is fails to be $(d-1)$-regular if and only if Z lies on a line.

Proof we induct on d. Let $Z' \subseteq Z$ be a length $(d-1)$ - subsche. Then

$$\cancel{\mathcal{I}_{Z'}} \quad 0 \to \mathcal{I}_Z \to \mathcal{I}_{Z'} \to k(p) \to 0$$

where p is a point $\mathcal{I}_{Z'}(d-1)$ is globally generated by induction.

We see $H^0(\mathcal{I}_{Z'}(d-1)) \twoheadrightarrow \mathcal{I}_{Z'} \otimes k(p) \to k(p)$

Since p is zero dimension the second is always surject. It follows

Excercise $H^1(\mathcal{I}_Z(d-1)) \hookrightarrow H^1 \mathcal{I}_{Z'}(d-1) = 0$.

p2.20 I ~~lea~~ Check the second part of the proposition.

Example 2.21. Let $x \in \mathbb{P}^n$ be a point. Let $\mathcal{M}_x$ be ideal sheaf of x. $\mathcal{M}_x^{a+1}$ is $(1+a)$-regular. Let Z be the scheme defined by $\mathcal{M}_x^{a+1}$. If $\mathcal{F}$ is m-regular then

$$H^0(\mathcal{F}(m+a)) \longrightarrow H^0 \mathcal{F}/\mathcal{M}_x^{a+1}\mathcal{F}(m+a)$$

is surjective.

Definition 2.22. Let $x \in \mathbb{P}^n$. Let $\mathcal{G}$ be a coherent sheaf on $\mathbb{P}^n$. We say the the sections of $\mathcal{G}$ separates a-jets of $\mathcal{F}$ at x, if

$$H^0(\mathcal{G}) \longrightarrow H^0(\mathcal{G}/\mathcal{M}_x^{a+1})$$

is surjective.

Remark 2.23. If length Z is a, then $\mathcal{I}_Z$ is a-regular.

~~Definition~~

Example 2.24. Let Z be closed subscheme of length a in $\mathbb{P}^n$. Then $\mathcal{I}_Z$ is a-regular

~~By~~ ... If $\mathcal{F}$ is m-regular, then

$$H^0(\mathcal{F}(m+a-1)) \longrightarrow H^0(\mathcal{F}(m+a-1)\otimes \mathcal{O}_Z)$$

is surjective. by 2.18.

Example 2.25 Let X be d general points in $\mathbb{P}^n$. Then the restriction

$H^0(\mathcal{O}_{\mathbb{P}}(m)) \to H^0(\mathcal{O}_X(m))$ is either injective or surjective. So if $\binom{m+n}{n} \geq d$

then $H^1(I_X(m)) = 0$ So $\operatorname{Reg}(\mathcal{I}_X) = O(d^{1/n})$.

Example 2.25.

Let $X \subseteq \mathbb{P}^2$ be 4 points. There are three possible resolutions for $\mathcal{I}_X$.

(i) Assume no three points X are collinear. $h^0(\mathcal{I}_X(2)) = 2$. We see X is a complete of two conics.

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-4) \longrightarrow 2\mathcal{O}_{\mathbb{P}}(-2) \longrightarrow \mathcal{I}_X \to 0.$$

$\mathrm{Reg}(\mathcal{I}_X) = 3$.

(ii) Three but not all four points are collinear. Then $\mathcal{I}_X$ has a resolution

$$0 \to \mathcal{O}(-3) + \mathcal{O}(-4) \longrightarrow \mathcal{O}(-2)^2 \oplus \mathcal{O}(-3) \longrightarrow \mathcal{I}_X \to 0$$

(ii) $\mathrm{Reg}(\mathcal{I}_X) = 3$.

(iii) All four points are collinear. Then X is a complete intersection of a line and a quartic.

$$0 \to \mathcal{O}(-5) \longrightarrow \mathcal{O}(-1) \oplus \mathcal{O}(-4) \longrightarrow \mathcal{I}_X$$

Then $\mathrm{Reg}(\mathcal{I}_X) = 4$.

Exercise

~~Example~~ 2.11 Let $d \geq 4$ be a positive integer. Consider the morphism from $\mathbb{P}^1 \xrightarrow{\varphi} \mathbb{P}^3$ given by $\varphi(s, t) = (s^d, s^{d-1}t, st^{d-1}, t^d)$. Set $C = \varphi(\mathbb{P}^1)$.

(i) Show φ is an embedding

(ii) Show that C has a $(d-1)$-secant line. Show $\mathcal{I}_C$ is not $(d-2)$-regular

(iii) Show the $\mathcal{I}_C$ is $(d-1)$-regular.