Representations of SL_2

$SL_2 = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid ad-bc=1 \}$

I. Examples of reps

Let V_n = space of homogeneous polynomials in x, y of degree n

e.g.
V_0 = constants $= \mathbb{k}$
V_1 = linear poly $= \{a_0 x + a_1 y\}$
V_2 = quadratic $= \{a_0 x^2 + a_1 xy + a_2 y^2\}$
etc

Let SL_2 act on V_n by:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \underset{V_n}{\underset{\Uparrow}{p(x,y)}} = p(ax+cy, bx+dy)$$

Exercise (easy) Check that this defines a representation

V_0 = the trivial rep
V_1 = the defining rep.

Exercise When $\mathbb{k} = \mathbb{C}$, V_n is an irreducible rep.

II. Subgroups of SL_2

Let $B = \{ \begin{bmatrix} * & \\ * & * \end{bmatrix} \} \subset SL_2$
$= \{ \begin{bmatrix} a & \\ c & a^{-1} \end{bmatrix} \}$

$U^+ = \{ \begin{bmatrix} 1 & u \\ & 1 \end{bmatrix} \} \subset SL_2$
$U^+ \simeq \mathbb{G}_a$.

Let $\mathbb{k}_n$ = the 1-dim'l B-rep on $\mathbb{k}$ where B acts by:

$$\begin{bmatrix} a & \\ c & a^{-1} \end{bmatrix} \cdot x = a^n x$$

Lemma ① Every irreducible B-rep is isomorphic to $\mathbb{k}_n$ for some $n \in \mathbb{Z}$

Proof: Exercise.

Lemma ② The set $U^+B = \{gh \mid g \in U^+, h \in B\}$ is dense (in the Zariski top) in SL_2.

Proof. $\begin{bmatrix} 1 & u \\ & 1 \end{bmatrix} \begin{bmatrix} a & \\ c & a^{-1} \end{bmatrix} = \begin{bmatrix} a+uc & a^{-1}u \\ c & a^{-1} \end{bmatrix}$

Every matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $d \neq 0$ arises in this way. So:

$$SL_2 \setminus U^+B = \{ \begin{bmatrix} p & -r^{-1} \\ r & 0 \end{bmatrix} \} \subset SL_2.$$

closed in the Zariski top.

U^+B is open.

To show that U^+B is dense explicitly:

Let $a \in k^\times$. Consider

$$\underset{\underset{U^+}{\cap}}{\begin{bmatrix} 1 & -ar^{-1} \\ & 1 \end{bmatrix}} \underset{\underset{B}{\cap}}{\begin{bmatrix} a & \\ r - a^{-1}rp & a^{-1} \end{bmatrix}}$$

$$= \begin{bmatrix} p & -r^{-1} \\ r - a^{-1}rp & a^{-1} \end{bmatrix} \in U^+B.$$

$$\lim_{a \to \infty} \left(\text{this} \right) = \begin{bmatrix} p & -r^{-1} \\ r & \end{bmatrix}$$

$\square$

III. Induction

G - alg gp

$H \subset G$ - alg subgp (closed in Zariski top)

Let V be an H-rep. $\pi : H \to GL(V)$.

Defn (Induction of representations)

Let $\operatorname{ind}_H^G V = \left\{ \begin{array}{l} \text{morphisms of var} \\ f : G \to V \end{array} \middle| \begin{array}{l} \forall g \in G,\ h \in H, \\ f(gh) = \pi(h^{-1}) f(g) \end{array} \right\}$

Let G act on $\operatorname{ind}_H^G V$ by:

if $g \in G$, $f \in \operatorname{ind}_H^G V$ then $(g \cdot f)(\underset{\underset{G}{\cap}}{g'}) = f(g^{-1} g')$.

WARNING: $\operatorname{ind}_H^G V$ may be ∞-dim'l but it's still a union of fin dim'l algebraic subreps.

Thm (Adjointness property of induction)

Let $H \subset G$ be a closed subgp

Let V_1 be a G-rep

V_2 be an H-rep.

Then there is a natural isom

$$\Theta : \operatorname{Hom}_H (V_1, V_2) \xrightarrow{\sim} \operatorname{Hom}_G (V_1, \operatorname{ind}_H^G V_2)$$

Proof Sketch

Define Θ as follows: for $\varphi : V_1 \to V_2$ let

$$\Theta(\varphi)(\underset{\underset{V_1}{\cap}}{v})(\underset{\underset{G}{\cap}}{g}) = \underset{\underset{V_2}{\cap}}{\varphi(g^{-1} \cdot v)},$$

Define Θ^{-1} as follows: for $\psi : V_1 \to \operatorname{ind}_H^G V_2$ let

$$\Theta^{-1}(\psi)(\underset{\underset{V_1}{\cap}}{v}) = \underset{\underset{\operatorname{ind}_H^G V_2}{\cap}}{\psi(v)}(1) \in V_2.$$

id elt of G.

Exercise These maps are indeed bijections, inverse to each other $\square$.

IV. Induction for SL_2

Let $I_n = \mathrm{ind}_B^{SL_2} \Bbbk_n$

$= \{ f : SL_2 \to \Bbbk \mid f(g\begin{bmatrix} a & \\ c & a^{-1}\end{bmatrix}) = a^{-n} f(g) \}$.

For $f \in I_n$ let

$\bar{f} = f|_{U^+} : U^+ \longrightarrow \Bbbk$

$U^+ = \{ \begin{bmatrix} 1 & u \\ & 1 \end{bmatrix} \} \simeq \mathbb{A}^1$

$\bar{f}$ is an element of the polynomial ring $\Bbbk[u]$.

Lemma ③ ⓐ The map $I_n \longrightarrow \Bbbk[u]$, $f \longmapsto \bar{f}$ is injective.

ⓑ $\forall f \in I_n$, $\lim_{a \to \infty} a^{-n} \bar{f}(au)$ exists.

Proof ⓐ By the definition of $\mathrm{ind}_B^{SL_2} \Bbbk_n$, for $f \in I_n$, $f|_{U^+}$ determines $f|_{U^+B}$. But U^+B dense in SL_2 so $f|_{U^+B}$ determines f. □

ⓑ Exercise. Related to the limit calculation from earlier □.

Prop. If $n<0$, $I_n = 0$.
If $n \geq 0$, $\dim I_n \leq n+1$.

Proof. ~~Lemma 3ⓑ implies that~~
Suppose $\bar{f} = c_0 + c_1 u + \cdots + c_k u^k$.
Then $a^{-n}\bar{f}(au) = a^{-n} c_0 + a^{-n+1} c_1 u + \cdots + a^{-n+k} c_k u^k$

$\lim_{a \to \infty}$ exists $\Rightarrow -n+k \leq 0$.

So $\deg \bar{f} \leq n$.

If $n<0$, $I_n = 0$.
If $n \geq 0$, $I_n \hookrightarrow \underbrace{\text{polynomials of deg} \leq n}_{\dim\ n+1}$. □

Thm. For $n \geq 0$, $V_n \simeq I_n$.

Proof. Define $\varphi : V_n \to I_n$ by

$\underset{V_n}{\underset{\cap}{\varphi(p)}}(\underset{SL_2}{\underset{\cap}{g}}) = \underset{V_n}{\underset{\cap}{(g^{-1} \cdot p)}}\Big|_{x=1, y=0} \in \Bbbk$.

Explicitly: if $p(x,y) = \sum a_i x^{n-i} y^i$ then $\overline{\varphi(p)} \in \Bbbk[u]$ is given by

$\left(\begin{bmatrix} 1 & u \\ & 1\end{bmatrix}^{-1} \cdot p\right)\Big|_{x=1, y=0} = \left(\sum a_i x^{n-i} (y-ux)^i\right)\Big|_{1,0}$

$= \sum (-1)^i a_i u^i$

This shows: φ injective. By dim counting φ is isom. □

V. Classification of irred reps

Let V be an irred SL_2-rep.

Regard it as a B-rep:

- if no longer irreducible, take a quotient by a proper subrep.
- repeat until you get an irred. rep.

We've shown: $\exists$ surjective (nonzero) map of B-rep

$$V \twoheadrightarrow \Bbbk_n \quad \text{for some } n \in \mathbb{Z}.$$

By adjointness thm, this corresp to some nonzero G-module map.

$$\varphi : V \hookrightarrow \mathrm{ind}_B^{SL_2} \Bbbk_n = I_n \simeq V_n$$

$\ker \varphi$ is a subrep of V. But V irred so $\ker \varphi = 0$. So φ is <u>injective</u>.

Its image is an irred. subrep of $I_n \simeq V_n$

Lemma ④ The image of φ in V_n contains the element x^n.

Proof Exercise.

Defn. Let L_n = smallest SL_2-subrep of V_n containing x^n.

Must have $L_n \subset \underbrace{\mathrm{im}\,\varphi}_{\text{irred.}}$.

So $L_n = \mathrm{im}\,\varphi$.

We've shown: part (a) of:

Thm a) Every irred SL_2-rep is isomorphic to L_n for some $n \geq 0$.

b) For $n \neq m$, $L_n \not\simeq L_m$, and each L_n is irreducible

We'll prove in Lecture 3.

In the case $\Bbbk = \mathbb{C}$: V_n is irreducible so $L_n = V_n$.

The irreducible reps of $SL_2(\mathbb{C})$ are the V_n.