Specialization of stable rationality

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17 June 2020, Fudan-SCMS Algebraic Geometry Seminar

This talk is based on ongoing project with John Christian Ottem.

Aim

specialization of stable birational types (N-Shinder)

$+$ tropical geometry

$\Rightarrow$ new examples of stably irrational complete intersections (quartic fivefold)

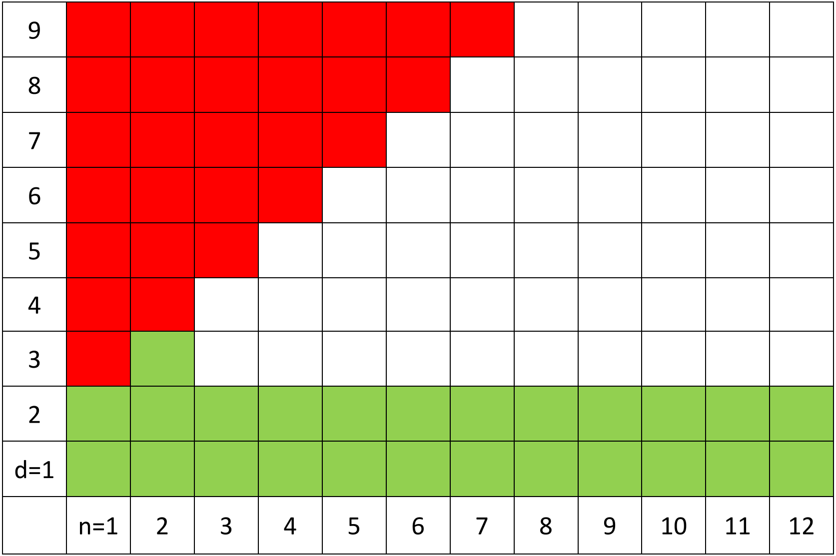
Definition

A complex algebraic variety X is called

- rational if it is birational to $\mathbb{P}^n_{\mathbb{C}}$, for some $n \geq 0$;
- stably rational if $X \times_{\mathbb{C}} \mathbb{P}^m_{\mathbb{C}}$ is rational, for some $m \geq 0$.

Stably rational $\not\Rightarrow$ rational in dimension ≥ 3 (Beauville, Colliot-Thélène, Sansuc & Swinnerton-Dyer 1985).

We will study stable rationality of very general hypersurfaces $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ of degree d.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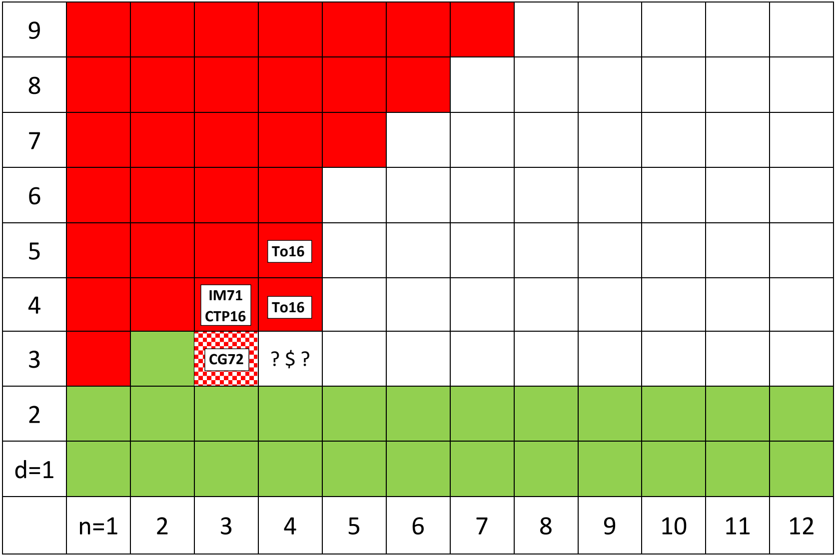
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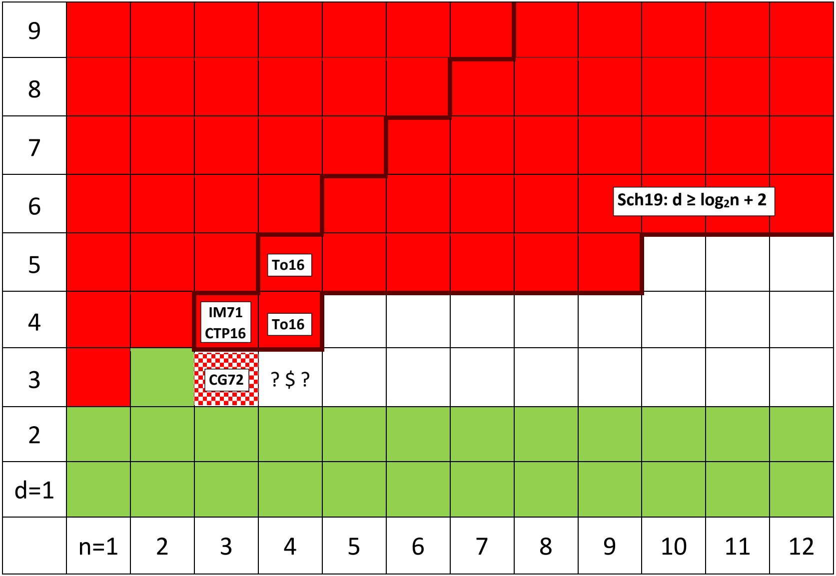
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Sch19: d ≥ log₂n + 2
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Question

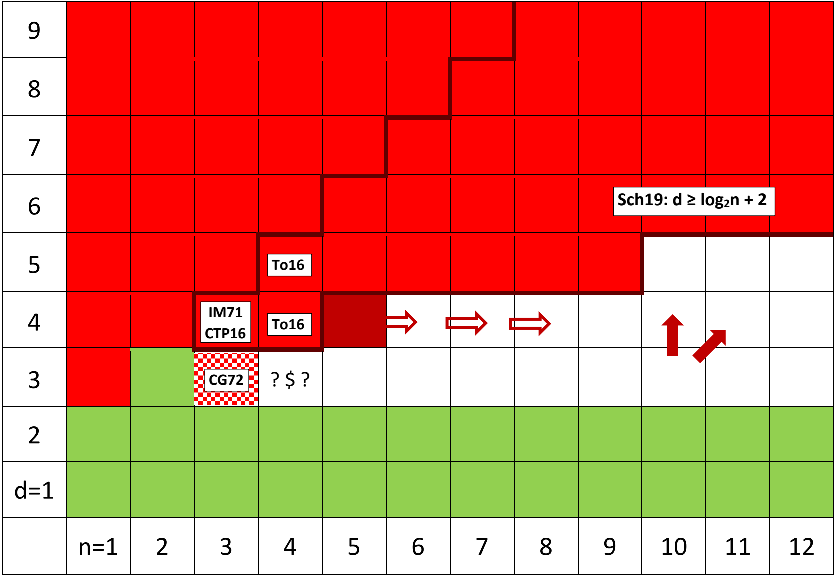
Are all smooth quartic hypersurfaces stably irrational?

We do not know any value of d such that a very general hypersurface of degree d is stably irrational in all dimensions.

Our results for hypersurfaces in $\mathbb{P}^{n+1}_{\mathbb{C}}$:

Theorem (N-Ottem)

1. A very general quartic fivefold is stably irrational (+ small improvement of Schreieder's bound).
2. Stable irrationality of a special quartic in dimension $n-1$ implies stable irrationality of very general quartics in dimension n.
3. Stable irrationality of a very general hypersurface of degree d and dimension n implies stable irrationality for $(d+1, n)$ and $(d+1, n+1)$.
4. Classification of stably irrational hypersurfaces in products of projective spaces up to dimension 5.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Some further results for stable irrationality of complete intersections:

Theorem (N-Ottem)

1. Very general $(2,3)$ complete intersection in $\mathbb{P}^6_{\mathbb{C}}$ (only open fourfold case besides cubic fourfold).
2. Very general $(3,3)$ complete intersection in $\mathbb{P}^7_{\mathbb{C}}$.
3. Schreieder-type bounds for complete intersections. Special case: very general complete intersection of r quadrics in $\mathbb{P}^{n+r}_{\mathbb{C}}$ for $n \geq 4$ and $r \geq n-1$

   (extends results by Hassett, Pirutka & Tschinkel and Chatzistamatiou & Levine)

These are special cases of more general specialization result for stably birational types.

Main ideas:

1. "Nearby cycles functor" for stable birational types (N-Shinder 2019); every strictly toroidal degeneration yields obstruction to stable rationality of geometric generic fiber.
2. Source of strictly toroidal degenerations: tropical geometry (Kushnirenko, Viro, Tevelev, Luxton & Qu).

Stable birational types

Let F be a field.

Definition

Two integral F-schemes of finite type X, Y are called stably birational if

$$X \times_F \mathbb{P}^{\ell}_F \sim_{\mathrm{bir}} Y \times_F \mathbb{P}^m_F$$

for some $\ell, m \geq 0$.

Note: stably rational = stably birational to $\operatorname{Spec} F$.

- SB_F = set of stable birational equivalence classes of integral F-schemes of finite type

- For every F-scheme X of finite type, we set

$$[X]_{\mathrm{sb}} = [X_1]_{\mathrm{sb}} + \ldots + [X_r]_{\mathrm{sb}} \quad \text{in } \mathbb{Z}[\mathrm{SB}_F]$$

  where $X_1, \ldots, X_r$ are the irreducible components of X.

- $\mathbb{Z}[\mathrm{SB}_F]$ has natural ring structure:

$$[X]_{\mathrm{sb}} \cdot [Y]_{\mathrm{sb}} = [X \times_F Y]_{\mathrm{sb}}$$

Specialization of stable birational types

Field of Puiseux series:

$$\mathbb{C}\{\{t\}\} = \bigcup_{m>0} \mathbb{C}((t^{1/m}))$$

Valuation ring:

$$\mathbb{C}\{\{t\}\}^o = \bigcup_{m>0} \mathbb{C}[[t^{1/m}]]$$

A $\mathbb{C}\{\{t\}\}^o$-scheme of finite type is called strictly toroidal if Zariski-locally, $\mathscr{X}$ admits a smooth morphism

$$\mathscr{X} \to \operatorname{Spec} \mathbb{C}\{\{t\}\}^o[M]/(\chi^m - t^q)$$

where

- M is a toric monoid,
- $q \in \mathbb{Q}_{>0}$,
- $m \in M$ such that $\mathbb{C}[M]/(\chi^m)$ is reduced.

Examples

1. Strictly semi-stable models: Zariski-locally smooth over

$$\operatorname{Spec} \mathbb{C}\{\{t\}\}^o[z_0, \ldots, z_n]/(z_0 \cdot \ldots \cdot z_n - t^q).$$

   Every smooth and proper $\mathbb{C}\{\{t\}\}$-scheme admits a strictly semi-stable proper $\mathbb{C}\{\{t\}\}^o$-model.

2. General $f \in \mathbb{C}[z_0, \ldots, z_n]$ homogeneous of degree d; general $g_1, \ldots, g_r \in \mathbb{C}[z_0, \ldots, z_n]$ homogeneous of degrees $d_1 + \ldots + d_r = d$. Then

$$\operatorname{Proj} \mathbb{C}\{\{t\}\}^o[z_0, \ldots, z_n]/(tf - g_1 \cdot \ldots \cdot g_r)$$

   is strictly toroidal.

If $\mathscr{X}$ is a strictly toroidal $\mathbb{C}\{\{t\}\}^o$-scheme, a stratum E of $\mathscr{X}_{\mathbb{C}}$ is a connected component of an intersection of irreducible components of $\mathscr{X}_{\mathbb{C}}$.

$\mathrm{codim}(E)$: codimension of E in $\mathscr{X}_{\mathbb{C}}$.

Theorem (N-Shinder 2019)

There exists a unique ring morphism

$$\mathrm{Vol}\colon \mathbb{Z}[\mathrm{SB}_{\mathbb{C}\{\{t\}\}}] \to \mathbb{Z}[\mathrm{SB}_{\mathbb{C}}]$$

such that for every strictly toroidal proper $\mathbb{C}\{\{t\}\}^o$*-scheme* $\mathscr{X}$*,*

$$\mathrm{Vol}([\mathscr{X}_{\mathbb{C}\{\{t\}\}}]_{\mathrm{sb}}) = \sum_{E} (-1)^{\mathrm{codim}(E)} [E]_{\mathrm{sb}}.$$

Proof relies on weak factorization and logarithmic geometry.

Vol sends $[\mathrm{Spec}\, \mathbb{C}\{\{t\}\}]_{\mathrm{sb}}$ to $[\mathrm{Spec}\, \mathbb{C}]_{\mathrm{sb}} \rightsquigarrow$ obstruction to stable rationality of $\mathscr{X}_{\mathbb{C}\{\{t\}\}}$.

If $\mathscr{X}$ is a smooth and proper $\mathbb{C}\{\{t\}\}^o$-scheme, then formula simplifies to

$$\mathrm{Vol}([\mathscr{X}_{\mathbb{C}\{\{t\}\}}]_{\mathrm{sb}}) = [\mathscr{X}_{\mathbb{C}}]_{\mathrm{sb}}.$$

Corollary

Stable rationality of geometric fibers specializes in smooth and proper families in characteristic zero.

More precisely:

Theorem (N-Shinder 2019)

S Noetherian $\mathbb{Q}$-scheme, $f: X \to S$ smooth and proper morphism. Then

$$\{s \in S \mid \text{geometric fiber of } f \text{ over } s \text{ is stably rational}\}$$

is a countable union of closed subsets of S.

Strengthened to rationality by Kontsevich & Tschinkel.

We now discuss a basic application where the obstruction does not vanish.

Theorem (Voisin 2015)

A very general quartic double threefold over $\mathbb{C}$ is stably irrational.

Proof. Artin-Mumford: $\exists$ stably irrational quartic double threefold $Y/\mathbb{C}$ with isolated ordinary double point singularities.

Putting the ramification divisor in a general pencil we get a double cover $\mathscr{Y} \to \mathbb{P}^3_{\mathbb{C}[t]}$ with special fiber Y and whose generic fiber is a smooth quartic double threefold.

Let $\mathscr{Y}' \to \mathscr{Y}$ be the blow-up at the singular locus of Y, and let $\mathscr{X}$ be the normalization of $\mathscr{Y}' \times_{\mathbb{C}[t]} \mathbb{C}\{\{t\}\}^o$.

Then $\mathscr{X}$ is strictly toroidal and its generic fiber $\mathscr{X}_{\mathbb{C}\{\{t\}\}}$ is a smooth quartic double threefold over $\mathbb{C}\{\{t\}\}$.

The special fiber $\mathscr{X}_{\mathbb{C}}$ has exactly one stably irrational stratum, namely, the strict transform of the Artin-Mumford threefold Y.

Thus

$$\sum_{E}(-1)^{\mathrm{codim}(E)}[E]_{\mathrm{sb}} \neq [\mathrm{Spec}\,\mathbb{C}]_{\mathrm{sb}} \quad \text{in } \mathbb{Z}[\mathrm{SB}_{\mathbb{C}}]$$

and $\mathscr{X}_{\mathbb{C}\{\{t\}\}}$ is not stably rational.

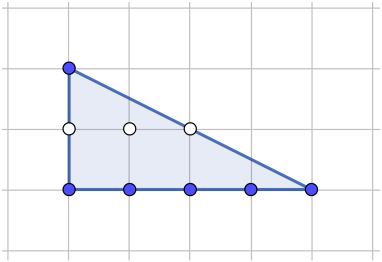
Our specialization theorem now implies that a very general double quartic threefold over $\mathbb{C}$ is not stably rational. $\square$

To apply this method to generate new applications, we need a rich source of strictly toroidal degenerations $\rightsquigarrow$ tropical geometry.

Tropical geometry: Newton polytopes

- $f = \sum_{m \in \mathbb{Z}^{n+1}} c_m x^m$ Laurent polynomial in $\mathbb{C}[\mathbb{Z}^{n+1}]$
- $\mathrm{Supp}(f) = \{m \in \mathbb{Z}^{n+1} \mid c_m \neq 0\}$
- Newton polytope Δ_f = convex hull of $\mathrm{Supp}(f)$ in $\mathbb{R}^{n+1}$
- $Z(f)$ = zero locus of f in $\mathbb{G}^{n+1}_{m,\mathbb{C}} = \mathrm{Spec}\,\mathbb{C}[\mathbb{Z}^{n+1}]$

Example: quartic double curve $y^2 = f_4(x)$.



Δ lattice polytope in $\mathbb{R}^{n+1}$

$\mathscr{P}$ polyhedral subdivision of Δ into lattice polytopes

Definition

We say that $\mathscr{P}$ is regular if there exists a convex piecewise affine function $\Delta \to \mathbb{R}$ whose domains of linearity are the maximal faces of $\mathscr{P}$.

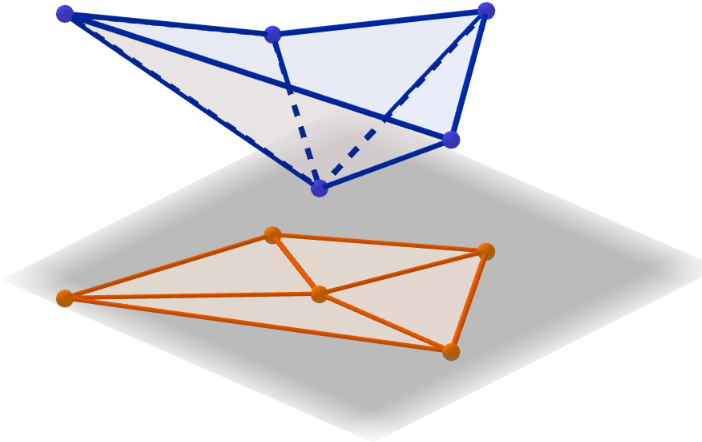


Figure: Regular subdivision of a polytope

Assume that $\mathscr{P}$ is regular.

Consider

$$\mathbb{Z}^{n+1} \cap \Delta \to \mathbb{C},\ m \mapsto c_m$$

such that $c_m \neq 0$ for every vertex in $\mathscr{P}$.

For every face δ of $\mathscr{P}$, set

$$f_\delta = \sum_{\mathbb{Z}^{n+1}\cap\delta} c_m x^m.$$

Then $\Delta_{f_\delta} = \delta$.

Assume that $Z(f_\delta)$ is smooth for all δ.

Main Theorem (N-Ottem)

If

$$\sum_{\delta \not\subset \partial\Delta} (-1)^{\mathrm{codim}(\delta)} [Z(f_\delta)]_{\mathrm{sb}} \neq [\mathrm{Spec}\,\mathbb{C}]_{\mathrm{sb}}$$

in $\mathbb{Z}[\mathrm{SB}_{\mathbb{C}}]$, then a very general hypersurface in $\mathbb{G}^{n+1}_{m,\mathbb{C}}$ with Newton polytope Δ is not stably rational.

Idea of proof: by our specialization theorem, it is enough to find one non-degenerate polynomial f over $\mathbb{C}\{\{t\}\}$ with Newton polytope Δ such that $Z(f)$ is not stably rational.

One can build from our tropical data a strictly toroidal $\mathbb{C}\{\{t\}\}^o$-model $\mathscr{X}$ such that $\mathscr{X}_{\mathbb{C}\{\{t\}\}}$ is a toroidal compactification of a non-degenerate hypersurface $Z(f)$ and the strata in $\mathscr{X}_{\mathbb{C}}$ are toroidal compactifications of the hypersurfaces $Z(f_\delta)$ with $\delta \not\subset \partial\Delta$.

Application: the quartic fivefold

Newton polytope of general quartic fivefold:

$$\Delta = \{(x_1, \ldots, x_6) \in \mathbb{R}^6_{\geq 0} \mid \sum_i x_i \leq 4\}$$

Subdivision $\mathscr{P}$: insert face

$$\delta_0 : \quad x_1 + x_2 + x_3 + x_4 + x_5 + 2x_6 = 4$$

(quartic double fourfold)

$\rightsquigarrow$ two isomorphic pieces δ_1, δ_2, intersecting along δ_0.

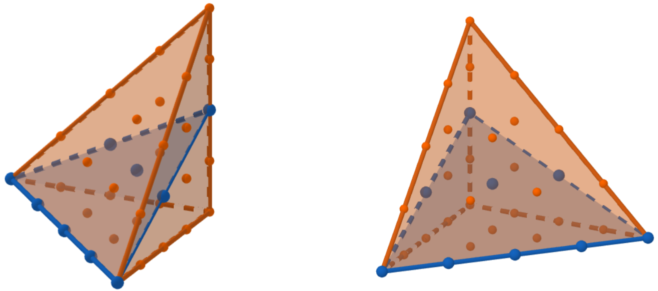


Figure: Newton polytope of general quartic surface containing Newton polytope of quartic double curve

If we choose symmetric coefficients c_m for f_{δ_1} and f_{δ_2}, then

$$\begin{aligned}\sum_{\delta\not\subset\partial\Delta}(-1)^{\mathrm{codim}(\delta)}[Z(f_\delta)]_{\mathrm{sb}} &= [Z(f_{\delta_1})]_{\mathrm{sb}}+[Z(f_{\delta_2})]_{\mathrm{sb}}-[Z(f_{\delta_0})]_{\mathrm{sb}}\\ &= 2[Z(f_{\delta_1})]_{\mathrm{sb}}-[Z(f_{\delta_0})]_{\mathrm{sb}}\\ &\equiv [Z(f_{\delta_0})]_{\mathrm{sb}} \mod 2\mathbb{Z}[\mathrm{SB}_{\mathbb{C}}].\end{aligned}$$

But δ_0 is the Newton polytope of a quartic double fourfold, so that $[Z(f_{\delta_0})]_{\mathrm{sb}}\neq[\mathrm{Spec}\,\mathbb{C}]_{\mathrm{sb}}$ (Hassett-Pirutka-Tschinkel 2019).