Let $X$ be projective variety and $\mathcal{L}$ be a very ample line bundle on $X$.

$$X \xhookrightarrow{\varphi} \mathbb{P}(H^0(\mathcal{L})) \simeq \mathbb{P}^r, \text{ where } r = h^0(\mathcal{L})-1.$$

Let $V = \cancel{S}\cdot H^0(\mathcal{L})$. and $S = Sym(V)$

$\simeq k[x_0, \dots, x_r]$.

Set $R_{\mathcal{L}} = R(X, \mathcal{L}) = \bigoplus_0^\infty H^0(\mathcal{L}^{\otimes m})$

Consid $R_{\mathcal{L}}$ as a $S$-module.

Supp

$$F_{\bullet} \quad F_r \to \dots \to F_1 \to F_0 \to R_{\mathcal{L}} \to 0$$

be its minimal graded free resolution.

~~A~~ Observe that $S_1 \simeq (R_{\mathcal{L}})_1 = H^0(\mathcal{L})$.

Def. 5.1. We say $|\mathcal{L}|$ ~~says~~ satisfies $N_0$ (or $|\mathcal{L}|$ is projectively normal) if $F_0 = S$.

i.e the natural map $S \to R_{\mathcal{L}}$ is surjective

Equivalently: $H^1(\mathcal{I}_X(m)) = 0$ for every $m$.

Def. 5.2 Inductively, we say $\mathcal{L}$ satisfies $N_p$ if. 2.

(1) $\mathcal{L}$ satisfies $N_{p-1}$.

(2) ~~$b_p$.~~ $\beta_{p,p+j}(R) = 0$ for $j \geq 2$. $\Leftrightarrow \mathrm{Tor}_p(R_L, k)_{p+j} = 0$ for $j \geq 2$.

$\Leftrightarrow$ (2)' $k_{p,j}(R) = 0$ for $j \geq 2$.

$\Leftrightarrow$ First $p$ steps of the resolution look le

$$F_{p+1} \to \oplus S(-p-1) \;---\; \to \oplus S(-2) \to S \to R_{\mathcal{L}}$$

$\Leftrightarrow$

Or $N_0$ + $\;---\; F_{p+1} \to \oplus S(-p-1) \;---\; \oplus S(-2) \to I_X \to 0$

~~$N_1 \Leftrightarrow$~~ $\mathcal{L}$ satisfies $N_1 \Leftrightarrow \mathcal{L}$ satisfies $N_0$

+ $I_X$ is generated by degree elements.

So

Excercise. Let $X$ be a projective variety. Suppose $\mathcal{L}$ is an ample and globally generated line bundle. Suppose $\varphi : \mathrm{Symm}(H^0(\mathcal{L})) \longrightarrow R(X, \mathcal{L})$ is surjective. Then $\mathcal{L}$ is very ~~ap~~ ample.

Lemma 5.1. Assume $\deg L \geq 2g+1$.
Then $\mathcal{O}_X(L)$ is very ample.

Proof. If $Z$ is length 2 subscheme of $C$.

$$H^0(\mathcal{O}_C(L)) \to H^0(L|_Z) \to H^1(L(-Z)) = 0$$

So $|L|$ is very ample.

Definition 5.2. A divisor $D$ is said to be $l$-very ample ($l \geq 0$), if

$$H^0(\mathcal{O}_X(D)) \longrightarrow H^0(\mathcal{O}_X(D)|_Z) \to 0 \text{ is surjed}$$

for every length $(l+1)$ subscheme of $C$.

Ex 5.2 If $\deg \mathcal{L} \geq 2g+p$ then $\mathcal{L}$ is $p$-very ample.

Lemma, 5.3. Suppose $x_1, \cdots x_m$ be $m$ general points on $C$. When $m \leq r-2$.

Then $\Lambda = \mathrm{Span}(x_1 \cdots x_n) \simeq \mathbb{P}^{m-1} \subseteq \mathbb{P}^r$.

Then $\Lambda \cap C = \{x_1 \cdots x_m\}$.

Proof. See Griffith and Harris.

Remark 5.4 ~~This m~~ Set $D = \sum_1^m x_i$.

5.3 $\longleftrightarrow$ $h^0(L(-D)) = (r+1)-m$ and $|L(-D)|$ does not have any base points.

Theorem 5.5. (Castelnuovo).

(i) If degree $L \geq 2g+1$, $\mathcal{O}_X$ is 2-regular with respect to $L$.

(ii) $|L|$ gives a projectn normu embeddy.

Proof (i) This follow from $H^1(\mathcal{O}_X(L)) = 0$

(ii) It is suffical to show

$H^0(L) \otimes H^0(L) \to H^0(L^2)$ is surject

~~Since~~ $H^0(L) \otimes H^0(L^m) \to H^0 L$

Suffl to si $H^0(L) \otimes H^0(L) \longrightarrow H^0(L^2)$ is surjecti

$$H^0(L) \otimes H^0(L^m) \to H^0(L^{m+1}) \text{ is surjecti}$$

by regularity. for $m \geq 2$.

Conside the exact sequ

$$0 \to M_L \to H^0(L) \otimes \mathcal{O}_X \to L \to 0$$

Obsere that $M_L \cong \varphi^* \Omega^1_{\mathbb{P}^r}(1)$.. So rank$(M_L) = r$.

We want approxmit $M_L$ by line bunls.

Chose $x_1 \cdots x_{r-1}$ general pond on $C$.

Set $D = \sum_1^{r-1} x_\lambda$

Consider diagram.

$$\begin{tikzcd} L^*(D) \arrow[r] \arrow[d] & H^0(L(-D)) \otimes \mathcal{O}_C \arrow[r, "\varphi"] \arrow[d] & L(-D) \arrow[d] \\ M_L \arrow[r] & H^0(L) \otimes \mathcal{O}_C \arrow[r] \arrow[d] & H^0 \cancel{(L)} \otimes L \arrow[d] \\ \oplus \mathcal{O}_C(-P_\lambda) & W \arrow[r, "\beta"] & L|_D \simeq \mathcal{O}_D \end{tikzcd}$$

$H^0(L(-D))$ is 2 dimen.

$\varphi$ is surject by Remark 5.4

Since $H^0(L)$ maps onto $H^1(L|_D)$.

We choose a chose a basis $s_1 \dots s_{r-1}$ for $W$ such $\beta(s_\lambda) = \delta_{P_\lambda}$. ~~Then~~ This implies the bottom is the correct form.

(ii) $\Longleftrightarrow H^1(M_L \otimes L) = 0$.

$$0 \to \mathcal{O}_X(D) \to H^1 M_L \otimes L) \to \oplus H^1(L(-P_\lambda)) \to 0$$

Rieman Roch $\Rightarrow$
$\cancel{r} \; r+1 = d+1-g \Rightarrow r-1 = d-1-g \geq$
$\cancel{2g+2} - g$.

So $h^1(\mathcal{O}_X(D)) = h^0(\omega_C(-D)) = 0$

Since $\deg D \geq g$ an $D$ is general.

Theorem 5.6 (Noether) Assume $C$ is nonhyperelliptic. Then $|\omega_C|$ gives a projectively normal embedd of $C$ in $\mathbb{P}(H^0(\omega_C)) = \mathbb{P}^{g-1}$.

Proof If $C$ is nonhyperelliptic, then $\omega_C$ is very ample. As in Theorem 5.5, we can choose $\{p_1, \dots p_{g-2}\}$ general points that spans a $\mathbb{P}^{g-3} \subseteq \mathbb{P}(H^0(\omega_C)) = \mathbb{P}^{g-1}$.

~~Then~~ $\omega$ Set $D = \sum_1^{g-2} p_i$. Then $h^0(\omega_C(-D)) = 2$ and span $(D) = \Lambda$, $\Lambda \cap C = \{p_1 \cdots p_{g-2}\}$. Then $\omega_C(-D)$ is globally genrated. As in 5.5, there is a diagrm

$$0 \to \omega_C^{*}(D) \to M_{\omega_C} \to \bigoplus_1^{g-2} \mathcal{O}_C(-p_i) \to 0 \quad -(5.61)$$

as in 5.5.

One checks $h^1(M_{\omega_C}^{\otimes \omega_C}) \leq g$.

$Tor_0(R_K, k)_2 = \ker(H^1(M_{\omega_C} \otimes \omega_C) \to H^0(\omega_C) \otimes H^1(\omega_C)) = 0$.

It follows that $Tor_2(R_K, k)_2 = 0$.

$Tor_0(R, k)_3 = 0$ is easier. $\omega_C$ is 3-regular.

So $Tor_0(R, k)_j = 0$ for $j \geq 4$.

Lemma 5.7 Assume $\omega_C$ satisfies $N_{p-1}$. $\omega_C$ satisfies $N_p$ if and only if $h^1(M_{\omega_C} \otimes \omega_C) \leq \binom{g}{p+1}$.

Proof. We would check $k_{p,2}(\omega_C) = 0$. One can $k_{p,q}(\omega_C) = 0$ for $q \geq 3$ using 5.6.1.

As before $H^1(\Lambda^p M_C \otimes \omega_C^{\otimes 2}) = (\operatorname{Ker} \partial_p)_{p+2}$.

~~Assu~~ As before

We consider

$$0 \to \Lambda^{p+1} M_{\omega_C} \otimes \omega_C \to \Lambda^{p+1} V \otimes \omega_C \to \Lambda^{p} M_\omega \otimes \omega_C^{\otimes 2} \to 0$$

One checks $H^1(\Lambda^p M_\omega \otimes \omega_C^{\otimes 2}) = 0$ since

~~$\operatorname{Tor}_{p-1}(R_\omega, k)_{p+2}$~~ $\omega_C$ satisfies $N_{p-1}$.

So $\operatorname{Tor}_p(R_{K_C}, k)_{p+2} = \operatorname{Ker}\big(H^1(\Lambda^{p+1} M_C \otimes \omega_C)$

$\to \Lambda^{p+1} V \otimes H^1(\omega_C) = \Lambda^{p+1} V \to 0\big)$

So $h^1(\Lambda^{p+1} M_{\omega_C} \otimes \omega_C) \geq \binom{g}{p+1}$.

$\operatorname{Tor}_p(R_{K_C}, k)_{p+2} = 0 \iff$ ~~the~~ $h^1(\Lambda^{p+1} M_\omega \otimes \omega_C) \leq \binom{g}{p+1}$

(hence equality).

The following is version of Petri's theorem on canonical curve. The proof is due to Green–Lazarsfeld

Theorem 5.8. Let $C$ be a nonhyperelliptic curve of genus $g$ $(g \geq 5)$. Assume there is a line bundle $A$ on $C$, such that $\deg A = g-1$, $h^0(A) = h^0(\omega_C \otimes A^{-1}) = 2$ and both $A$ & $\omega_C \otimes A^{-1}$ are globally generated. Then ~~Choose~~ ~~K~~ $\omega_C$ satisfies $N_1$.

Proof. choose $D \in |A|$ be a general divisor. Write $D = p_1 + \cdots + p_{g-1}$. Consider the diagram

$$\begin{tikzcd}
& & 0 \arrow[d] & 0 \arrow[d] \\
& \omega_C^{*} \otimes A \arrow[d] & H^0(\omega_C \otimes A^{-1}) \otimes \mathcal{O} \arrow[d] & \omega_C \otimes A^{-1} \arrow[d] \\
0 \arrow[r] & M_\omega \arrow[r] \arrow[d] & H^0(\omega_C) \otimes \mathcal{O}_C \arrow[r] \arrow[d] & \omega_C \arrow[d] \\
0 \arrow[r] & \mathcal{E}_D \arrow[r] \arrow[d] & W_D \otimes \mathcal{O} \arrow[r] \arrow[d] & \omega_C|_D \arrow[d] \\
& 0 & 0 & 0
\end{tikzcd}$$

when $W_D = H^1(\omega_C) \big/ H^1(\omega_C \otimes A^{-1})$ is a 

$g-2$ dimensional vector space

rank $\mathcal{E}_D$ is $g-2$.

Notice our assumption implies that

~~Span~~ $\operatorname{Span}(D) = \Lambda_D \simeq \mathbb{P}^{g-3}$.

$\Lambda_D \cap C = \{p_1, \cdots, p_{g-1}\}$.

(*) If $\Sigma \subsetneq \{p_1, \cdots, p_{g-1}\}$ and $|\Sigma| = m+1$

$\operatorname{span}(\Sigma) = \Lambda_\Sigma \simeq \mathbb{P}^m$

and $\Lambda_\Sigma \cap C = \Sigma$.

Write $D = \underbrace{(x_1 + x_2)}_{E} + \underbrace{(x_3 + \cdots + x_{g-1})}_{D'}$

Consider the diagram

$$\begin{tikzcd}
 & 0 \arrow[r] & \mathcal{O}_C(-x_1-\dots-x_r) \arrow[r] & H^1(\omega_C(-D'))\otimes\mathcal{O}_C \arrow[r] \arrow[d] & \omega_E \arrow[r] & \dots \\
0 \arrow[r] & \mathcal{E}_D \arrow[r] \arrow[d] & & W_D\otimes\mathcal{O}_C \arrow[r] \arrow[d] & \omega_C|_D \arrow[d] & \\
0 \arrow[r] & \mathcal{E}_{D'} \arrow[r] & & W_{D'}\otimes\mathcal{O}_C \arrow[r] & \omega_C|_{D'} \arrow[d, "\varphi"] & \\
 & & & & \mathcal{E}_{D'}\otimes A &
\end{tikzcd}$$

$$\mathcal{E}_{D'} \simeq \bigoplus_{j}^{g-1} \mathcal{O}_C(-P_j)$$

$$\begin{tikzcd}
 & \omega_C(-x_1-\dots-x_r) \arrow[d] & & & \\
0 \arrow[r] & \mathcal{E}_D\otimes\omega_C \arrow[r] \arrow[d] & \Lambda^2 M_{\omega_C}\otimes\omega_C \arrow[r] & \Lambda^2\mathcal{E}_D \arrow[r] \arrow[d] & 0 \\
 & \mathcal{E}_{D'}\otimes\omega_C & & \Lambda^2\mathcal{E}_{D'}\otimes\omega_C \arrow[d] & \\
 & & & 0 &
\end{tikzcd}$$

Now a straightforward calculation

show that $h^1(M_C\otimes\omega_C) \le \binom{g}{2}$.

Exercise 5.7. Do the calculation.

Remark The existence of $A$ with the desired properties is saying $C$ is not trigonal and $C \not\cong$ plane curve of degree 5 by Mumford - Martens's theory See [ACGH]

Before we go on let me introduce a few more invariants

Definition 5.10 A line bundle $\mathcal{L}$ on a projective variety $X$, is said to be $l$-very ample for some non-negative integer $l$ if $H^0(\mathcal{L}) \twoheadrightarrow H^0(\mathcal{L}|_Z)$ for every length $(l+1)$ - closed subscheme $Z$ in $X$.

Remark 5.11. 0-very ample $\longleftrightarrow$ $\mathcal{L}$ is globally generated
1-very ample $\longleftrightarrow$ $\mathcal{L}$ is ample.

Lemma 5.12. Let $C$ be a smooth curve of genus $g$. $\omega_C$ fails to be $l$-very ample, if and only if there is a divisor $D$ of length $(l+1)$ such that $h^0(\mathcal{O}_C(D)) \geq 2$.

Proof Consider $H^0(\overset{\omega_C}{\mathcal{L}}) \xrightarrow{\varphi} H^0(\overset{\omega_C}{\mathcal{L}}|_D)$
$\rightarrow H^1(\omega_C(-D)) \xrightarrow{\psi} H^1(\omega_C)_k \rightarrow 0$.

Observe that $\mathrm{Cok}\,\varphi = \ker \psi$.

So ~~ker~~ $\mathrm{Cok}\,\varphi \neq 0 \iff \ker \psi \neq 0$

$\iff h^1(\omega_C(-D)) = h^0(\mathcal{O}_C(D)) \geq 2$.

Definition 5.13. The gonality of $C$ is defined as:

$$\mathrm{gon}(C) = \min\{d \mid \exists \text{ a line bundle } \mathcal{L} \text{ of degree } d,\ h^0(\mathcal{L}) \geq 2\}$$

$= \min\{d \mid$ there is a finite map of degree $d$ from $C$ to $\mathbb{P}^1\}$.

Proposition 5.14. $\omega_C$ is $p$-very ample if and only if $\mathrm{gon}(C) \geq p+2$.

$(g \geq 5)$.

Suppose $\mathcal{L}$ is a line bundle on $C$

(Motivated by Clifford theorem), one define the Clifford index of $\mathcal{L}$ to be

$$\begin{aligned}
\mathrm{Cliff}(\mathcal{L}) &= \deg \mathcal{L} - 2r(\mathcal{L}) \\
&= \deg \mathcal{L} - 2(h^0(\mathcal{L}) - 1) \\
&= (g+1) - h^0(\mathcal{L}) - \cancel{h^1(\mathcal{L})}\, h^0(\omega_C \otimes \mathcal{L}^{-1}) \\
&= \mathrm{Cliff}(\omega_C \otimes \mathcal{L}^{-1}).
\end{aligned}$$

The Clifford index of $C$ is define as

$$\mathrm{Min}\{\mathrm{Cliff}(\mathcal{L}) \mid h^0(\mathcal{L}) \geq 2,\ h^0(\omega_C \otimes \mathcal{L}^{-1}) \geq 2\}$$

~~We h~~ The following are two of the conjectures from the 80's:

So $R(X, \mathcal{L})$ and $R(C, \omega_C)$ have the same graded Betti numbers.

So enough to prove $R(X, \mathcal{L})$ satisfies $N_p$ $(X, \mathcal{L})$ satis $N_p$

for $p = [\frac{g}{2}] - 1 = m - 1$.

ie $\mathrm{Tor}_{m-1}(R(X,L), k)_{m+2} = 0$.

This computed by $V = H^0(X, \mathcal{L})$

$$\cancel{0 \to \Lambda^m V \otimes \to \Lambda}$$

$$\to \Lambda^m M_{\mathcal{L}} \otimes \mathcal{L} \to \Lambda^m V \otimes \mathcal{L} \to \Lambda^{m-1} M_{\mathcal{L}} \otimes \mathcal{L}^2 \to$$

Since $H^1(\Lambda^m V \otimes \mathcal{L}) = 0$.

$\mathrm{Tor}(R(X, L), k)_{m+2} = 0 \iff H^1(\Lambda^m M_{\mathcal{L}} \otimes \mathcal{L}) = 0$

It is much easier to prove a vanish theorem than an inequality.

**Conjecture 5.15.**

Green's conjecture on syzygies of canonical curves.

$(C, \omega_C)$ satisfies $N_{p-1}$ but not $N_p$ if and only if $\mathrm{Cliff}(C) = p$

**Theorem 5.16** (Voisin, Schreyer).
$(C, \omega_C)$ satisfies $N_2 \iff \mathrm{Cliff}(C) \geq 3$.

**Thm (5.17)** (Green - Lazarsfeld)
If $\mathrm{Cliff}(C) = p$, then $(C, \omega_C)$ does not satisfy $N_p$.

**Theorem 5.18** If $C$ is general curve, then $\mathrm{Cliff}(C) = [\frac{g}{2}] - 1$ & further $\mathrm{Cliff}(C)$ is computed by a line bundle $\mathcal{L}$ of degree $[\frac{g}{2}] + 1$ and $h^0(\mathcal{L}) = 2$.

Then $\mathrm{gon}(C) = [\frac{g}{2}] + 1$

Theorem (Voisin, 2002 for even genus 2005 for odd genus). Green's Conj for the syzygy of ~~&~~ canonical curves holds for a general curve.

Conjecture Green-Lazarsfeld's gonality conjecture.

Let $B$ be globally generated line bundle on $C$. Let $B$ be a very ample line ~~Set $R(C(X,$~~ and $L$ be a very ample line bundle on $C$

Set $R(C(X, L); \overset{\omega_C}{B}) = \bigoplus_m H^0(\overset{\omega_C}{B} \otimes L^{\otimes m})$.

For $\deg L >> 0$

$\dim \mathrm{Tor}_p(R(C(X,L), B, k))_{p+1} = k_{p,1}(R(X, L, B)) = 0$

$\Leftrightarrow \omega_C$ is $p$-very ample $\Leftrightarrow \mathrm{gon}(C) \geq p+2$.

The conjecture was settled by Ein-Lazarsfeld

in 2015.

~~M.~~ Remark.

$h^1(\bigwedge^{p+1} M_{\omega_C} \otimes \omega_C) > \binom{g}{p+1}$ is a

closed condition on $M_g$ = moduli space

of smooth curves of genus $g$

So to prove the conjecture we only

need to ~~exp~~ exhibit a single curve $C$

of genus $g$ ~~such~~ such that

$p = [\frac{g}{2}] - 1$, such that $(C, \omega_C)$ satisfies

~~$h^1(M_{\omega_C}$~~ $N_{p-1}$. $\iff$ $h^1(\bigwedge^p M_\omega \otimes \omega_C) = \binom{g}{p}$.

Idea (Voisin)

Consider $X$ a K3 surface

(i.e. $K_X = \mathcal{O}_X$ and $h^1(\mathcal{O}_X) = 0$).

and ~~$\mathcal{L}$~~ $H$ a "very" ample divisor on $X$

Assume $\operatorname{Pic} X \cong \mathbb{Z} \cdot [H]$.

$C \in |H|$

$2g_C - 2 = H^2 \qquad \mathcal{O}_X(H)|_C = \omega_C$.

by the adjunction formula.

Let me only discuss the case $g_C = 2m$

So $H^2 = 4m - 2$.

~~Then~~ So $\omega_C$ is very ample.

and $|\omega_C|$ is projectively normal.

$H^1(\mathcal{L}^m) = 0$ for all $m$. by $m=0$ since $X$ is K3 for $m \neq 0$, then it follows from Kodaira-vanishing theorem.

~~So R~~

$0 \to R(X, \mathcal{L})(-1) \xrightarrow{f} R(X, \mathcal{L}) \to R(C, \omega_C) \to 0$

is exact.

Thm (Lazarsfeld)
~~Theorem~~ If $C \in |\mathcal{L}|$ is general, then $C$ is Petri general.

For our purpose, it means there is a line bundle $A$ degree $m+1$ on $C$ such that $h^0(A)=2$, $A$ is globally generated, and $H^0(A) \otimes H^0(\omega_C \otimes A^{-1}) \xrightarrow{\mu_A} H^0(K_C)$ is ~~an~~ an isomorphism. ($h^0(\omega_C \otimes A^{-1}) = m$)

$\mu$ is called the Petri map.

($C$ is Petri general, mean $\mu_B$ is injective for every line bundle $B$ on $C$).

Consider

$$0 \to \mathcal{F}(C,A) \to 2 H^0(A) \otimes \mathcal{O}_X \to A_C \to 0$$

and

$$0 \to 2\mathcal{O} \to \mathcal{E}(C,A) = \mathcal{F}(C,A)^* \to \omega_C \otimes A^{-1} \to 0$$

So $h(\mathcal{E}) = m+2$.

$\mathcal{E}$ is a rank 2 vector bundle

& $\Lambda^2\mathcal{E} \simeq \mathcal{L}$ $\quad c_2(\mathcal{E}) = m+1$.

Idea is to explore the univer

zero scheme $\mathcal{E}$.

Set

$$0 \to M_{\mathcal{E}} \to H^0(\mathcal{E}) \otimes \mathcal{O}_X \to \mathcal{E} \to 0$$

~~Set~~ $\forall$ $x \in X$

$$M_{\mathcal{E}} \otimes k(x) = \{ s \in H^0(\mathcal{E}) \mid s(x) = 0 \}.$$

Set $Q_{\mathcal{E}} =$

$$Z = \mathbb{P}(Q_{\mathcal{E}}) \subseteq X \times \mathbb{P}(H^0(\mathcal{E})^\vee)$$

is the universal zero scheme of $\mathcal{E}$.

In Voisin's paper she study the

geometry $\mathbb{P}(H^0(\mathcal{E})^\vee) \subseteq \mathrm{Hilb}^{m+1}(X)$.

In the recent paper gave a simple proof

of Voisin Thm by directly exploring the geometry

of $Z \subseteq X \times \mathbb{P}(H^0(\mathcal{E})^\vee)$.