Nielsen Realization Problem For Sphere twists in 3-manifolds

jt with Bena Tshishiku

Lei Chen (陈蕾)

at GGT@Fudan

1) What's Nielsen Realization Problem?

1932 Nielsen:

Whether finite subgp of $MCG(S_g)$ has realization?

$$\begin{tikzcd} \mathrm{Diff}^+(M) \arrow[r] & \pi_0(\mathrm{Diff}^+(M)) =: MCG(M) \arrow[l, dashed, bend left, "?"] \\ & G \arrow[u, phantom, "\vee"] \end{tikzcd}$$

G finite

$\dim M = 2$

1934 Nielsen: if G is cyclic, then $\exists$ realization.

1983 Kerckhoff: Any finite subgp of $MCG(S_g)$ has realization.

Pf: $MCG(S_g) \curvearrowright \mathrm{Teich}(S_g) \cong \mathbb{R}^{6g-6}$, m_{Teich} $\Rightarrow$ $\forall G \underset{\text{finite}}{<} MCG(S_g)$ G has fixed pt in $\mathrm{Teich}(S_g)$

isometry

2) A generalized version of Nielsen Realization

a general subgp of $MCG(S_g)$

C^2-regularity

1989 Morita: $g \geq 18$, $MCG(S_g)$ has no realization $\mathrm{Diff}^2(S_g)$

Pf idea: $\exists$ section

$$\begin{tikzcd} \mathrm{Diff}^2(S_g) \arrow[r, "\pi_g"] & MCG(S_g) \arrow[l, bend left, dashed] \end{tikzcd} \Rightarrow \pi_g^*: H^*(MCG(S_g);\mathbb{Z}) \to H^*(\mathrm{Diff}^2(S_g);\mathbb{Z}) \text{ injective.}$$

Morita construct MMM classes $K_i \in H^{2i}(MCG(S_g);\mathbb{Z})$

$\pi_g^*(K_3) = 0$, $K_3 \neq 0$ (Bott Vanishing Thm) $\square$

Thurston: $H^*(\mathrm{Homeo}^\delta(S_g);\mathbb{Z}) = H^*(MCG(S_g);\mathbb{Z})$

2007 Markovic: $g \geq 6$, $\mathrm{Homeo}^+(S_g) \to MCG(S_g)$ has no section!

idea: shadowing lemma, turn a random realization into something nice!

pseudo-Anosov, Anosov

3) More technique to prove this result.

2009 Franks - Handel $g \geq 3$ C^1- pf — Thurston Stability

2013 Bestvina - Church - Souto $g \geq 3$ C^1 — Milnor-Wood Inequality

2016 Salter - Tshishiku $g \geq 3$ C^1-symmetry + Thurston Stability
(Braid gp version)

2018 Chen $g \geq 2$ $\mathrm{Homeo}(S_g)$, building on Markovic (Braid gp) > fixed point argument.

2020 Chen - Salter $g \geq 2$, elementary pf.

2019 Chen - Markovic : Torelli gp has no realization $g \geq 6$.
(building on Markovic, + Fixed point theory)

4) Geometric interpretation of NR!

$$\{\pi_1(B) \to MCG(S_g)\}/\sim \quad \overset{1:1}{\Longleftrightarrow} \quad \left\{ \begin{array}{c} S_g \to E \\ \downarrow \\ B \end{array} \right\} / \text{isomorphism}$$

$$\underline{\{\pi_1(B) \to \mathrm{Diff}^{+}(S_g)\}/\sim} \quad \overset{1:1}{\Longleftrightarrow} \quad \left\{ \begin{array}{c} S_g \to E \\ \downarrow \\ B \end{array} \text{ with a flat structure} \right\}$$

$$\longrightarrow \quad \begin{array}{c} S_g \to \tilde{B} \times S_g / \pi_1(B) \\ \downarrow \\ \tilde{B}/\pi_1(B) = B \end{array}$$

Q: If $\dim M \leq 5$, is there an S_g-bundle over M without a flat structure. $\left(K_3 \in H^6(MCG(S_g); \mathbb{Z}) \right)$

5) Higher dim

$\dim M \leq 3$ $\quad \pi_0(\mathrm{Diff}) = \pi_0(\mathrm{Homeo})$ $\qquad \dim M \geq 4$ $\quad \pi_0(\mathrm{Diff}(M)) \neq \pi_0(\mathrm{Homeo})$

(Donaldson, Ruberman)

$\underline{\dim M = 3}$ Now we focus on finite group.

<u>Thurston Geometrization</u>: Prime decomposition + Torus decomposition

$\Longrightarrow$ geometric pieces with 8 types.

(almost all known)

<u>Today</u>: Twist(M) twist subgp

6) Define $\mathrm{Twist}(M)$

$S \hookrightarrow M$ embedded 2-sphere

τ_S spheretwist $\quad \mathrm{supp}(\tau_S) \subseteq S \times [0,1] \subseteq M$

$$\tau_S(x, t) = (\rho_t(x), t) \qquad \rho_t \in \underline{\mathrm{Diff}(S^2)}$$

$$\pi_1(\mathrm{Diff}(S^2)) \overset{\text{Smale}}{=} \pi_1(SO(3)) = \mathbb{Z}/2$$

$$\tau_S^2 \simeq \mathrm{id} \Rightarrow \tau_S \in MCG(M)$$

is order 2.

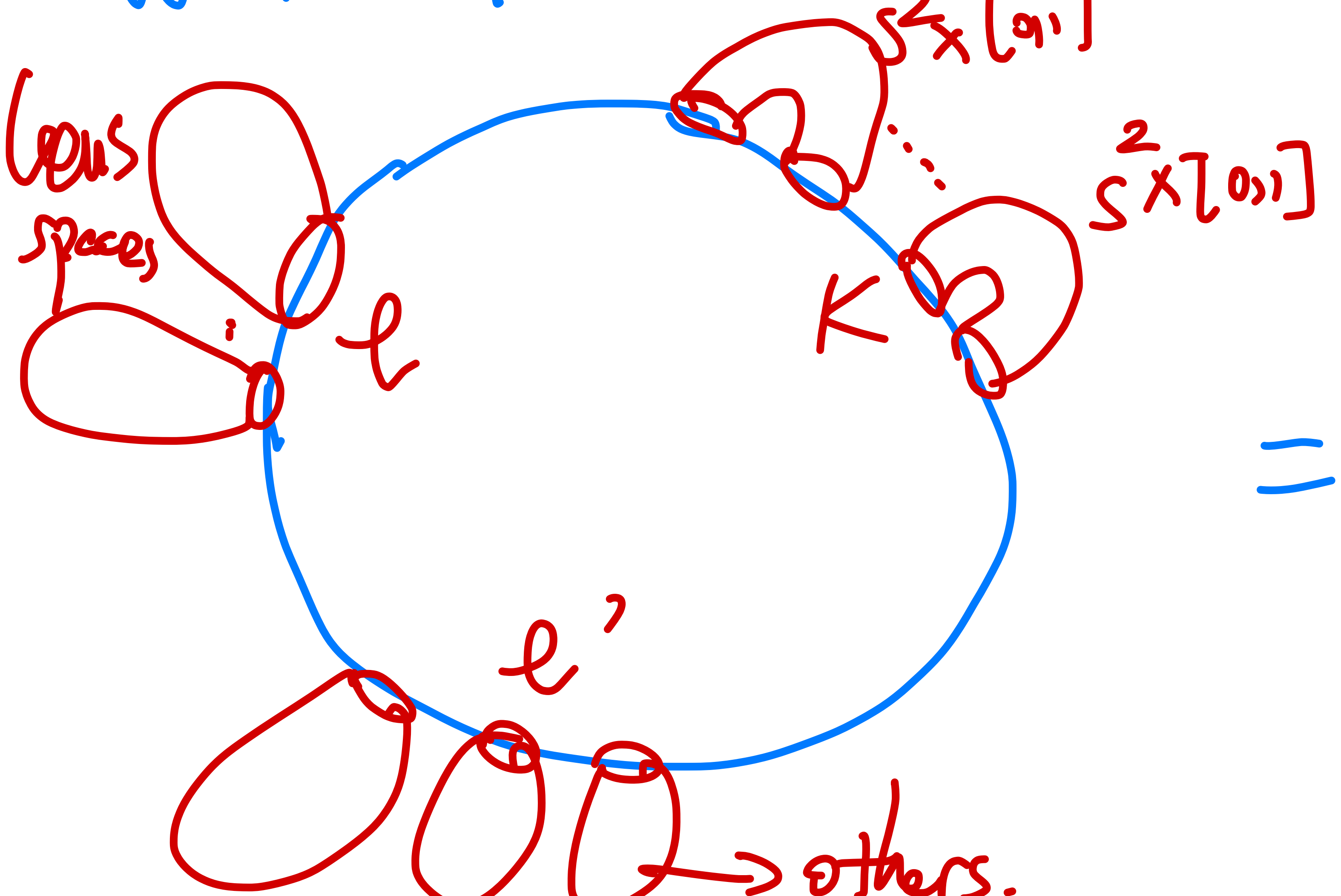
$$\underline{\mathrm{Twist}(M)} := \langle \tau_S \mid S \hookrightarrow M \rangle \subseteq MCG(M)$$

is a finite group.

7) Main Result

> **Thm** If $G < \mathrm{Twist}(M^3)$,
> G realizable $\iff$ G cyclic & M = connected sum of lens spaces.

What's $\mathrm{Twist}(M^3)$?

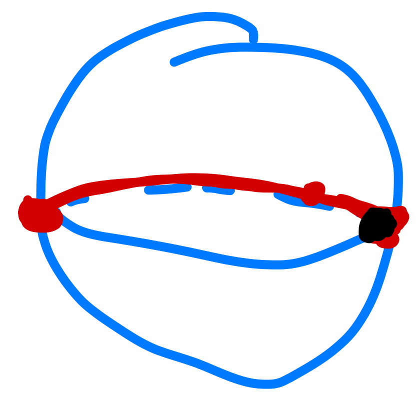


$$\mathrm{Twist}(M) = \begin{cases} (\mathbb{Z}/2)^{K} & \text{if } \ell' = 0 \\ (\mathbb{Z}/2)^{K+\ell'-1} & \text{if } \ell' \neq 0 \end{cases}$$

8) How to realize?

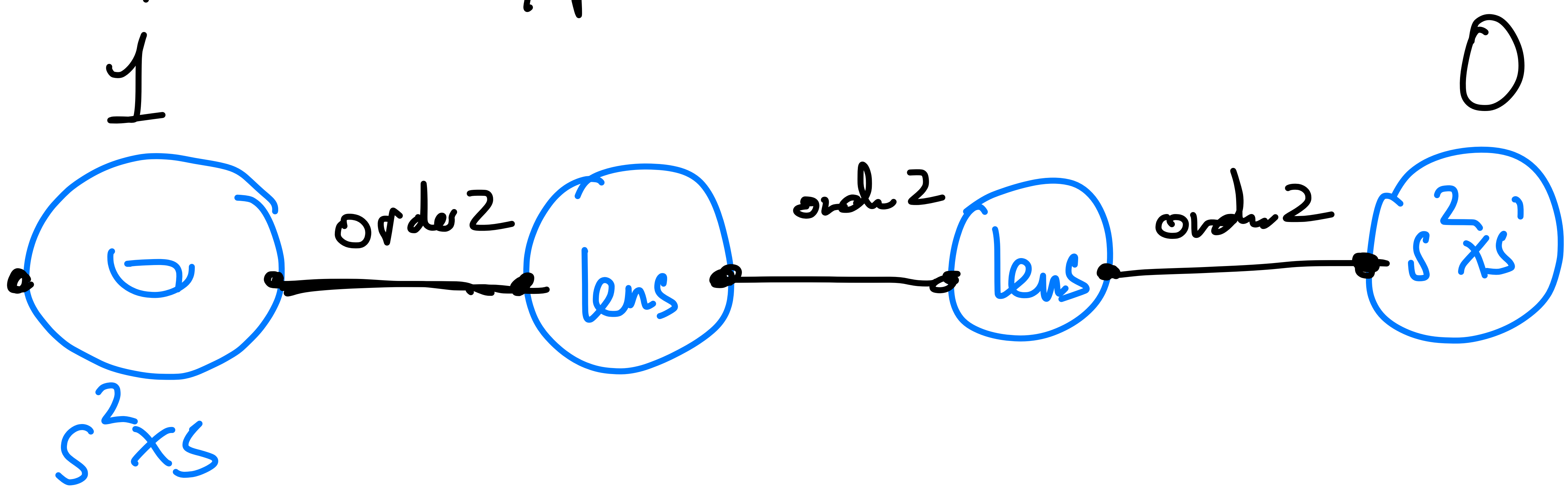
$\underline{\underline{S^2 \times S^1}}$: $(x, \theta) \longmapsto (R_{\theta,\pi}(x), \theta)$

2 fixed points.



nontrivial loop in $SO(3)$

lens space $S^3/\mathbb{Z}_p \circlearrowleft \mathbb{Z}/2$. with 2 fixed points.



9) The Obstruction Part.

Meeks-Yau Thm:

traditional sphere Thm: $\pi_2(M) \neq 0 \Rightarrow M$ contains an embedded sphere, $\exists$ a collection $\mathcal{S}$ s.t

$$M - \mathcal{S} = \text{irreducible pieces.}$$

Equivariant Sphere Thm: Finite gp $G \curvearrowright M$

$\exists$ a collection of sphere $\mathcal{S}$ s.t $\mathcal{S}$ is G-invariant.

$$M - \mathcal{S} = \text{irreducible pieces.}$$

Step 1: $\underset{\text{(up to conjugation)}}{\pi_1(M)\ \pi_2(M)} \curvearrowleft G$ trivially $\quad \left(G \leq \mathrm{Twist}(M)\right)$

Lemma: G preserves each element in $\mathcal{S}$.

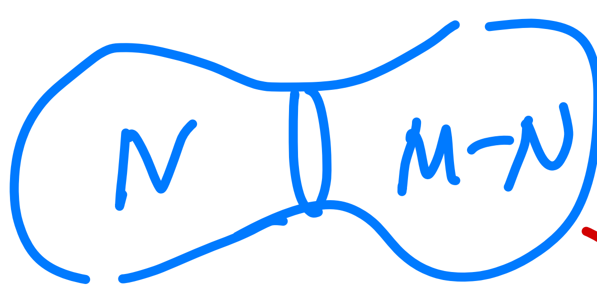
Step 2: N one component of $M - S$

$\hat{N} = N \cup \cup B^3$ $\quad \hat{N}$ closed $\circlearrowleft G$

$p \in$ one of the balls, is the $\{0\}$.

$G \curvearrowright \hat{N}$ $\quad G$ fixes p.

Lemma 2: $G \curvearrowright \pi_1(\hat{N}, p)$ is trivial.



$\pi_1(M) = \pi_1(N) * \pi_1(M-N)$

$\circlearrowleft G$

Lemma 3: $\tilde{\hat{N}} \circlearrowleft G$

contractible or S^3

$$\begin{array}{ccc} \tilde{\hat{N}} & & \\ \downarrow & & \uparrow \text{lift.} \\ \hat{N} & \circlearrowleft & G \end{array}$$

$\because \exists$ a fixed point p $\Rightarrow \exists$ lift to $\tilde{\hat{N}}$ with fixed pt $\tilde{p}$.

$G \curvearrowright \tilde{\hat{N}}$

Lemma 3: $G \circlearrowleft (\hat{N}, p)$ trivial on $\pi_1(\hat{N}, p)$

$\Rightarrow G \circlearrowleft (\tilde{\hat{N}}, \tilde{p})$ commutes with Deck gp $\pi_1(\hat{N}, p)$.

$\Rightarrow \underline{G} \times \pi_1(\hat{N}, p) \circlearrowleft (\tilde{\hat{N}}, \tilde{p})$

$\mathrm{Fix}(G) \setminus \overbrace{\text{1-dim manifold}}^{\text{connected}} \circlearrowleft \pi_1(\hat{N}, p)$

~~freely properly discontinuous~~ on $\mathbb{R}$ or S^1

$\Downarrow$

$\underline{\pi_1(\hat{N}, p)} = \underline{\mathbb{Z}}$ or $\mathbb{Z}/\underline{p}$

cyclic

$\Downarrow$

lens space!