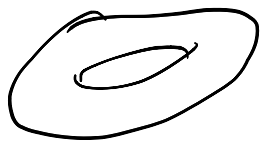
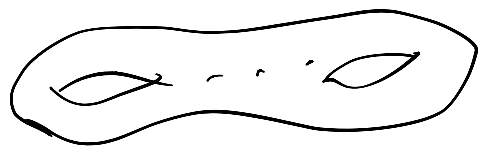
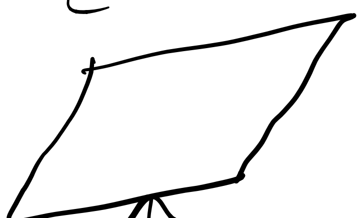
$\mathcal{H}$: Poincaré upper half plane.

$$\mathcal{H} = \{x+iy \mid y>0\} \subseteq \mathbb{C}$$

$\mathbb{CP}^1$, , 

$c>0$ $\quad$ $c=0$ $\quad$ $c<0$.



$\mathbb{C}$ $\quad$ $\mathcal{H}$.

$$\mathcal{H} \circlearrowleft GL_2(\mathbb{R})^+ \quad \frac{dx\,dy}{y^2}.$$

$$GL_2(\mathbb{R})^+ = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R}) \,\middle|\, \det>0 \right\} \ni \gamma$$

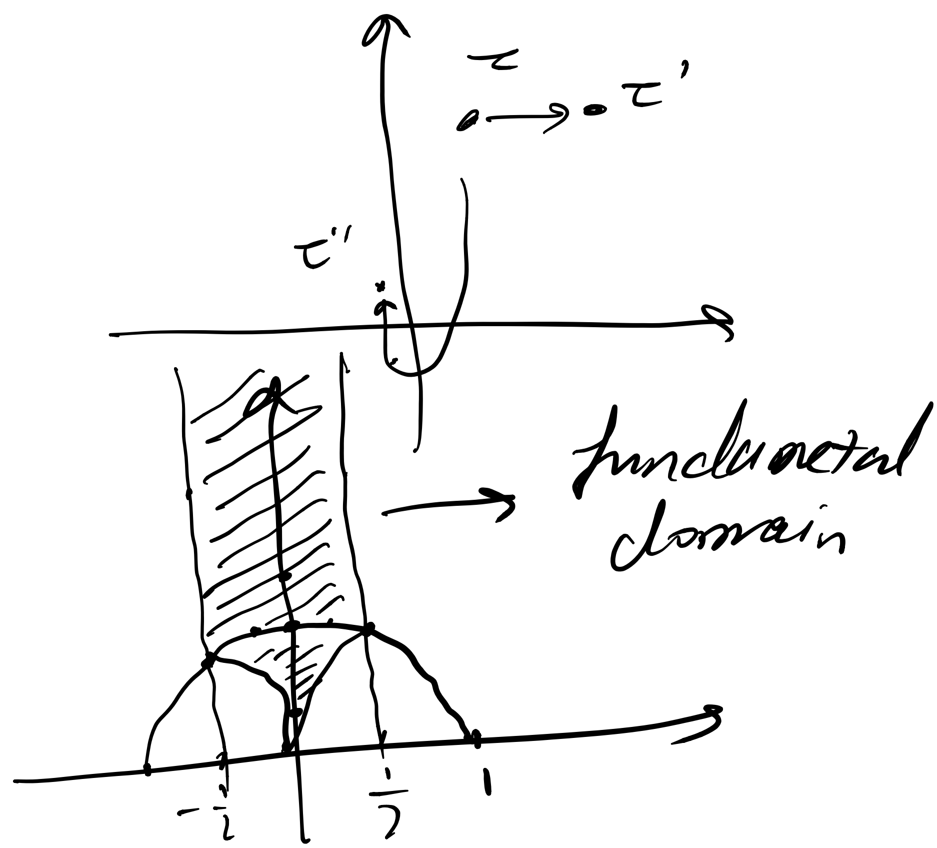
$$\gamma.\tau = \frac{a\tau+b}{c\tau+d}. \qquad \tau\in\mathcal{H}$$

Def: modular group

$SL_2(\mathbb{Z}) \subseteq GL_2(\mathbb{R})^+$.

Lem: $SL_2(\mathbb{Z})$ is gen. by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

$\tau \mapsto \tau + 1$, $\tau \mapsto -\frac{1}{\tau}$.



Pf: $SL_2(\mathbb{Z})$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Step 1. $\begin{pmatrix} a & b \\ c \pm d & d \end{pmatrix} \dashrightarrow$

$|c| < |d|$

Step 2. $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ switch c & d.

$|d|$ decreases

$\cdots\cdots$ $d = \pm 1$

$\begin{pmatrix} \pm 1 & 0 \\ ? & \pm 1 \end{pmatrix}$ $\begin{pmatrix} \pm 1 & ? \\ 0 & \pm 1 \end{pmatrix}$

$\square$.

Def: $k \in \mathbb{Z}$ (weight)

A mero function $f : \mathcal{H} \to \mathbb{C}$ is

__weakly modular of wt k__

if $\forall \gamma \in SL_2(\mathbb{Z})$.

$$f(\gamma(\tau)) = (c\tau+d)^k f(\tau)$$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

e.g. $k=0$. $f(\gamma(\tau)) = f(\tau)$.

f descent to a $\overbrace{\text{fcn}}$

on $SL_2(\mathbb{Z})\backslash\mathcal{H}$, meromorphic

$k=2$: $d\tau$ diff. form on $\mathcal{H}$.

$\gamma^* d\tau = ?$

$$d\left(\frac{a\tau+b}{c\tau+d}\right) = \frac{d\tau}{(c\tau+d)^2}$$

$$\gamma^* d\tau = (c\tau+d)^{-2}\, d\tau$$

f: weakly modular of wt 2.

$f d\tau$ diff form on $\mathcal{H}$

$$\gamma^*(f\, d\tau) = f\, d\tau$$

$f\,d\tau$ descends to a mero diff form on $SL_2(\mathbb{Z})\backslash\mathcal{H}$

f weakly mod. wt k

Take $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

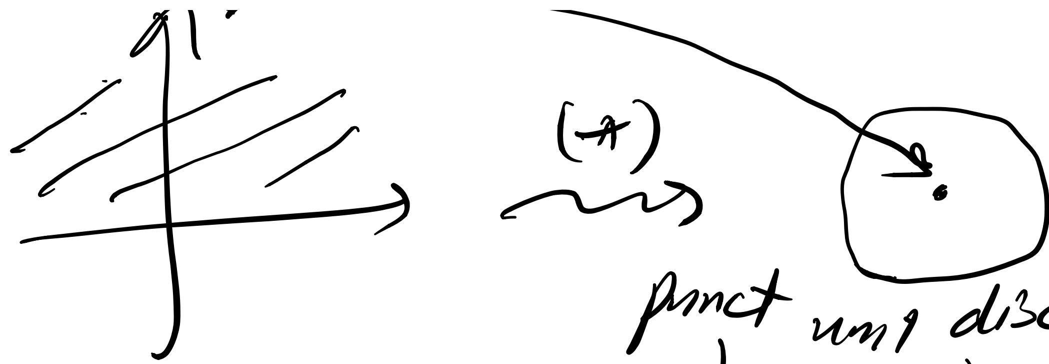
$$\boxed{f(\gamma(\tau)) = f(\tau)} = f(\tau+1)$$

f factors through

$$\tau \mapsto e^{2\pi i \tau} \quad (*)$$

f induces a meromorphic function on the image of $\mathcal{H}$ under $(*)$

$e^{i\infty}$



punct unit disc

$\{q \in \mathbb{C} \mid |q|<1\}$, $q \neq 0$.

$$f(\tau)=\sum_{n=-\infty}^{\infty} a_n q^n \qquad q=e^{2\pi i \tau}$$

Def: Say $f:\mathcal{H}\to\mathbb{C}$ is a **modular form of wt k** if

(1) f is holomorphic on $\mathcal{H}$

(2) f is holomorphic at ∞. (i.e., $f(\tau)=\sum_{n=0}^{\infty} a_n q^n$).

(3) f is weakly modular of wt k.

Denote by

$M_k(SL_2(\mathbb{Z}))$ to be the set of mod. forms of wt k.

which is a complex space.

$$\boxed{\bigoplus_{k\in\mathbb{Z}} M_k(SL_2(\mathbb{Z}))}$$

$f_i \in M_{k_i}(SL_2(\mathbb{Z}))$ $i=1,2$

$f_1 \cdot f_2 \in M_{k_1+k_2}(SL_2(\mathbb{Z}))$.

→ graded commutative complex alg.

Prop: $M_k(SL_2(\mathbb{Z})) = 0$

(1) if k is odd

(2) if $k < 0$

(1) $\gamma = \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}}_{\text{acts trivially on } \mathcal{H}} \in SL_2(\mathbb{Z})$.

$f(\tau) = (-1)^k f(\tau), \forall \tau$

$\Rightarrow f(\tau) \equiv 0$ if k is odd.

<u>Def</u>: Say a modular form f of wt k is a cusp form if f vanishes at ∞

i.e. $f(\tau) = \sum_{n=1}^{\infty} a_n q^n$.

$S_k(SL_2(\mathbb{Z})) \subseteq M_k(SL_2(\mathbb{Z}))$

subspace of cusp forms.

<u>Rmk</u>: $f \in M_k(SL_2(\mathbb{Z}))$

$g \in S_{k'}(SL_2(\mathbb{Z}))$

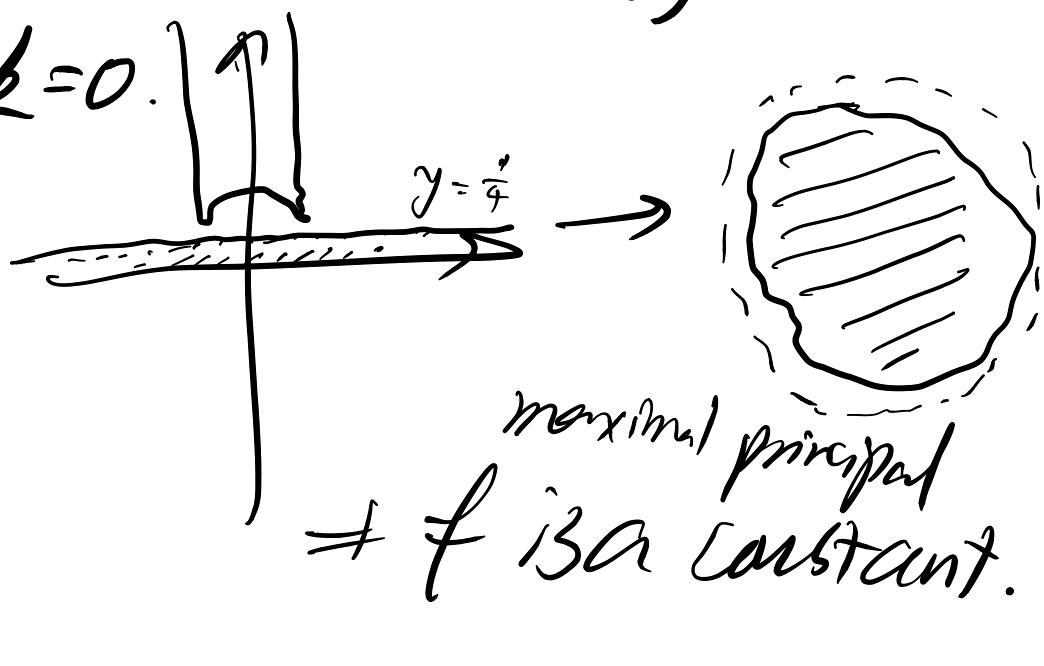
$f \cdot g \in S_{k+k'}(SL_2(\mathbb{Z}))$

$\Rightarrow \bigoplus_{k \in \mathbb{Z}} S_k(SL_2(\mathbb{Z}))$

is a graded module over the graded ring $\bigoplus M_k(SL_2(\mathbb{Z}))$

e.g. $M_k(SL_2(\mathbb{Z}))$

$k=0$.



maximal principal

$\Rightarrow f$ is a constant.

$\Rightarrow M_0(SL_2(\mathbb{Z})) = \mathbb{C}.$

$k > 0$ integer

$$G_k(\tau) := \sum_{\substack{(c,d) \\ \in \mathbb{Z}^2 \setminus \{(0,0)\}}} \frac{1}{(c\tau + d)^k}$$

$k \geq 4$. $G_k(\tau)$ is abs. convergent, and $G_k(\tau)$ is a modular form of wt k.

<u>Fact</u>: $k \geq 4$

$$G_k(\tau) = \boxed{2\zeta(k)} + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$$

Riemann zeta fcn.

$$\sigma_{k-1}(n) = \sum_{0 < d \mid n} d^{k-1}.$$

$$G_k(\tau) \in M_k(SL_2(\mathbb{Z})) \setminus S_k(SL_2(\mathbb{Z}))$$

$$\begin{aligned}\Delta(\tau) &= (60G_4)^3 - 27(140G_6)^2 \\ &= (2\pi)^{12} q + \cdots \\ &\in S_{12}(SL_2(\mathbb{Z}))\end{aligned}$$

例: $M_k = 0$ 对 $k<0$.

$f \in M_k$, for $k<0$.

$$f^{12}\cdot\Delta^{-k} \in S_0(SL_2(\mathbb{Z})) = 0$$

$$S_0(SL_2(\mathbb{Z})) \subset M_0(SL_2(\mathbb{Z})) = \mathbb{C}.$$

$$\Downarrow$$

$$f = 0.$$

E.g. $j = 1728 \dfrac{G_4^{\,3}}{\Delta}$.

meromorphic, weakly modular of wt 0. $SL_2(\mathbb{Z})\backslash\mathcal{H}$

$$j(\tau) = \frac{1}{q} + 744 + 196884q + \cdots$$

$$SL_2(\mathbb{Z})\backslash\mathcal{H} \xleftrightarrow{1-1} \left\{\begin{array}{c}\text{complex tori}\\ \mathbb{C}/\Lambda\end{array}\right\}\Big/\cong$$

$$\|$$

$$\mathbb{C} \xleftarrow{\ j\ } \left\{\begin{array}{c}\text{complex ellip}\\ \text{curve}\end{array}\right\}\Big/\cong$$

j-invariant.

Generalize $SL_2(\mathbb{Z})$:
congruence subgrp.

$\forall N>0$

$$SL_2(\mathbb{Z}) \longrightarrow SL_2(\mathbb{Z}/N\mathbb{Z}).$$

whose kernel is denoted by $\Gamma(N)$.

$\Gamma(N) \subset SL_2(\mathbb{Z})$ of finite index

Def: A congruence subgp is a subgp $\Gamma \leq SL_2(\mathbb{Z})$ s.t. $\Gamma \geq \Gamma(N)$ for some $N>0$.

<u>Rmk</u>: If Γ is a congruence sub. then Γ is of finite index in $SL_2(\mathbb{Z})$

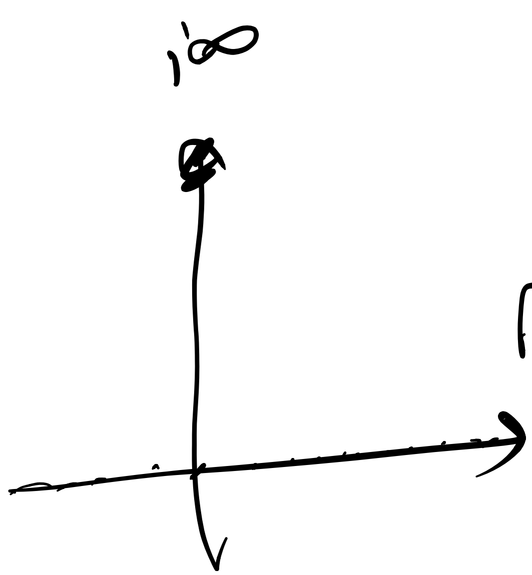
However, the converse is wrong.

q-exp. $\Gamma \ni \begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix}$ for some $h > 0$.

f is weakly modular of level Γ.

$$f(\tau+h) = f(\tau)$$

$$f(\tau) = \sum_{n=-\infty}^{\infty} a_n q^{\frac{n}{h}}$$



$$q \mapsto e^{2\pi i \tau}$$

$$e^{2\pi i \frac{\tau}{h}}$$

$\Gamma \subseteq$ $SL_2(\mathbb{Z}) \cdot i\infty$

$= i\infty \cup \mathbb{Q}$.

cusp.

f is modular if
f is holomorphic on $\mathcal{H}$
and every cusp.