Lecture IV

Castelnuovo-Mumford regularity is the basic measure of complexity of an ideal. Therefore it is of interest to establish bounds on this invariant. As Bayer and Mumford stressed in their influential article [BM], there is ~~strong~~ striking dichotomy between nice an arbitrary ideal. For instance, the regularity of ~~a~~ smooth projective varieties turns out to be at worst linear in the natural parameters, even optimal statements are not always known. ~~There~~

For instance, Koh ([K])
constructed an ideal I_r generated
$22r-3$ quadrics in a polynomial
ring $22r-\$$ variables, such that
$Reg(I_r) \geq 2^{2^{r-1}}$. based on an earlier construction by Mayr-Meyer
Eisenbud and Goto conjecture (~1983) if I_X is a homogen
prime ideal in $k[x_0, \cdots x_n]$, then
$Reg(I_X) \leq \deg X - \operatorname{Codim} X + 1$. — (4.1)
In a recent paper, McCullough-Peeva
show that in fact there can not be
a polynome bound ~~in term de~~
~~the regularity by a polynome bound~~
in ~~term of degree of~~ X.
Theorem 4.2 (McCullough - Peeva (~2018))
There does not exist a polynome $Q(x)$
such that $Reg(I_X) \leq Q(\deg X)$ for
every non-degenerate projective variety.

A modified Conjecture is the following

If X is a smooth nondegenerate variety of degree d in $\mathbb{P}^n$

Then $\mathrm{Reg}(\mathcal{I}_X) \le d - e + 1$.

where $e = \mathrm{Codim}\, X$. Recall $X \subseteq \mathbb{P}^n$ is nondegenerate, if X is not contained in a hyperplane. i.e. $H^0(\mathcal{I}_X(1)) = 0$.

We know several simple examples at the boundary bound.

43. (i) X is a smooth hypersurface of degree d (≥ 2) in $\mathbb{P}^n$. The $\mathcal{I}_X = \mathcal{O}_{\mathbb{P}^n}(-d)$

$\mathrm{Reg}(\mathcal{I}_X) = d$ and $\mathrm{Codim}(X) = e = 1$

$d = d - 1 + 1 = d$.

43 (ii) Next we consider the rational normal curve of degree n in $\mathbb{P}^n$.

$$\mathbb{P}^1 \xrightarrow[\varphi]{(s^n,\, s^{n-1}t,\, \dots,\, st^{n-1},\, t^n)} \mathbb{P}^n \qquad C = \varphi(\mathbb{P}^1)$$

$$U \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \quad \otimes\, \mathcal{O}^n$$

$$\|$$

$$0 \to \bigoplus_1^n \mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{A} \mathcal{O}_{\mathbb{P}^1}^{n+1} \xrightarrow{(s^n, \dots, t^n)} \mathcal{O}_{\mathbb{P}^1}(n)$$

$$\begin{array}{cc} & n \\ \underset{n+1}{A} = & \begin{pmatrix} t & 0 & & 0 \\ -s & t & & \vdots \\ 0 & -s & & \vdots \\ \vdots & \vdots & & t \\ 0 & 0 & & -s \end{pmatrix} \end{array}$$

We can consider Γ_φ, the graph of φ in $\mathbb{P}^1 \times \mathbb{P}^n$.

One check that Γ_φ is defined by n bilinear forms

$$\begin{aligned} tx_0 - sx_1 &= 0 \\ tx_1 - sx_2 &= 0 \\ &\vdots \\ tx_{n-1} - sx_n &= 0 \end{aligned} \qquad \operatorname{codim}_{\mathbb{P}^1\times\mathbb{P}^n} \Gamma_\varphi = n$$

So it is a complete intersection.

Set $W = \mathbb{P}^1 \times \mathbb{P}^n \xrightarrow{\pi_2} \mathbb{P}^n$.

$$\pi_1 \downarrow \quad \mathbb{P}^1$$

Write $\mathcal{O}_W(a,b) = \pi_1^* \mathcal{O}_{\mathbb{P}^1}(a) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^n}(b)$.

Then $\mathcal{O}_{\Gamma_W}$ is resolve by the Koszul

Complex :

$$0 \to \Lambda^n \mathcal{U} \otimes \mathcal{O}_W(-n,-n) \to \cdots \quad \Lambda^2 \mathcal{U} \otimes \mathcal{O}_W(-2,-2) \to \mathcal{U} \otimes \mathcal{O}_W(-1,-1) \to \mathcal{O}_W \to \mathcal{O}_{\Gamma_W} \to$$

Apply π_{2*}, we obtain.

$$0 \to \Lambda^n \mathcal{U} \otimes H^1(\mathcal{O}_{\mathbb{P}^1}(-n)) \otimes \mathcal{O}_{\mathbb{P}^n}(-n) \to \Lambda^{n-1} \mathcal{U} \otimes H^1(\mathcal{O}_{\mathbb{P}^1}(-n-1)) \otimes \mathcal{O}_{\mathbb{P}^n}(-n+1) \cdots$$

$$\to \Lambda^2 \mathcal{U} \otimes H^1(\mathcal{O}_{\mathbb{P}^1}(-2)) \otimes \mathcal{O}(-2) \xrightarrow{\quad} \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_C \to 0$$

$$\searrow \mathcal{I}_C \nearrow$$

$$0 \nearrow$$

One see $\mathcal{I}_C$ is 2-regular. and

I_C is general $\binom{n}{2}$ quads.

One check $\hat{C}$ = the cone over C

$$0 \to \Lambda^n \mathcal{U} \otimes H^n(\mathcal{O}_{\mathbb{P}^1}(-n)) \otimes \cancel{\mathcal{O}_{\mathbb{P}^{n-1}}}\ S(-n) \to \cdots \to \Lambda^2 \mathcal{U} \otimes S(-2) \to S \to S/I_{\hat{C}}$$

— (4.3)

The above complex is called an Eagon-Northcott complex. 6

One sees that $\operatorname{Proj\,dim} S/I_C = n-1$

Hence $\operatorname{depth}(S/I_C) = 2$ by the Auslander-Buchbaum formula. So we have a regular sequence of length 2.

Let l_1, l_2 be two general elements of S_1. Set $\bar{S} = S/\langle l_1, l_2 \rangle$

One checks $\dim(I_C \otimes \bar{S})_2 = \binom{n}{2} = \bar{S}_2$

So $(I_C \otimes \bar{S}) = M_{\bar{S}}^2 = (\bar{S}_+)^2$.

Proposition 4.4 To see that if $M \subseteq k[x_1, \ldots, x_n]$ is the maximal homogeneous ideal, then M^2 is resolved by an Eagon-Northcott complex. Also I_C, I_C and $\bar{M}^2$ are 2-regular.

A nondegenerate projective variety X of codim e in $\mathbb{P}^n$, if $\deg X = e+1$.

One can show that $\hat{X}$ is Cohen Macaulay. If $\Lambda \subseteq \mathbb{A}^{n+1}$ is a general linear subspace of codim $(n+1)-e$, then $\cancel{S}$ $(S/I) \otimes S_\Lambda \simeq S_\Lambda/M^2$. Hence the graded Betti number of I_X is computed by the Eagon-Northcott complex.

Exercise 4.4. Check the above assertion.

Example 4.5 $\varphi: \mathbb{P}^1 \xrightarrow{(s^d, s^{d-1}t, st^{d-1}, t^d)} \mathbb{P}^3$ $(d \geq 5)$ has a $d-2$-secant. So $\operatorname{Reg}(I_X) \geq d-1$.

Gruson-Lazarsfeld-Peskine say $\operatorname{Reg}(I_X) \leq d-1$. So $\operatorname{Reg}(I_X) = d-1$. This is yet another extremal example

Eisenbud-Goto Conjecture is true

for smooth projective variety, if $\dim X \le 2$

($\dim = 1$, Gruson-Lazarsfeld-Peskine, $\dim X = 2$, Lazarsfeld

$\dim X = 3$ $\operatorname{Reg}(X) \le \deg X - e + 2$ (Kwak), $\dim = 4$

$\operatorname{Reg}(X) \le \deg X - e + 6$ (Kwak).

M

Let I be a homogen ideal in $k[x_1 - x_n]$. 10

Let

Let $F.$ be the minimal resolut of I_x

$$\cdots F_2 \to F_1 \to F_0 \to I \to 0$$

Write $F_1 = \oplus S(-j)^{\beta_{1,q}}$.

$\beta_{1,j} = \dim \mathrm{Tor}_1(I_x, k)_j$

$\beta_{0,j}$ = # of minimal generat of I_x of dgee j.

β_{1j} = # of of minimal first syzy of dgree j

For regularity it is convenient to set

$k_{p,q} = \beta_{p,p+q}$.

$$B(I): \quad \begin{array}{|lll} k_{02} & k_{12} & k_{22} \quad \cdots \\ k_{01} & k_{11} & k_{21} \quad \cdots \\ k_{00} & k_{10} & k_{20} \quad \cdots \\ \hline \end{array}$$

$m = \mathrm{Reg}(I) = \mathrm{Max}\{j \mid \beta_{i,i+j} \neq 0 \text{ for some } i\}$

$\qquad\qquad = \mathrm{Max}\{j \mid k_{i,j} \neq 0 \text{ for some } i\}$. mth row $\neq 0$

$\quad \|$

height of the Betti table. $(m+j)$-st row $= 0$ for $j \geq 1$.

Suppose pth column $\neq 0$. but $(p+1)$-th column $= 0$

$\Longleftrightarrow F_p \neq 0 \quad F_{p+1} = 0$.

$\Longrightarrow p = \mathrm{Proj\,dim}(I)$.

We see the width of the Betti table measures the projectn dimen.

Example ~~4.~~ C is a nonhyperelliptic curve of genus g $(g \geq 3)$. Then K_C is very ample

$$C \xrightarrow{|K_C|} \mathbb{P}(H^0(K_C)) = \mathbb{P}^{g-1}$$

$$0 \to I_C \to \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_C \to 0$$

$$H^1(\mathcal{O}_C(1)) = H^1(\omega_C) \neq 0 \qquad \text{for } j \geq 2$$

$$H^1(\mathcal{O}_C(j)) = H^1(\omega_C^j) \simeq H^2(\mathcal{I}_C(j)) = 0$$

So ~~I~~ One checks that $\mathcal{I}_C$ is ~~3~~ 4-regular.

The fact that $H^1(\mathcal{I}_C(j)) = 0$ for all j is ~~called~~ a theorem of Noether. We'll describe that next time. So the height of Betti table is ≤ 4.

[MP]
J. McCullough and J. Peeva,
Counterexamples to the Eisenbud - Goto regularity conjecture, J. of the Amer. Math. Soc. Vol 31, No 2, (2018)
473–493.

I would like to outline their construction without addressing the technical details.

$I = \langle f_1 \cdots f_m \rangle \subseteq S$

where f_i's are minimal generators of I

$\operatorname{Maxdeg}(I) \overset{=}{\text{def}} \operatorname{Max}_{i} \{\deg f_i\}$.

It is known that $\operatorname{Reg}(I_x) \le (2 \max\deg I_x)^{2^{p-2}}$

On the other hand, Koh (based on an earlier construction of Mayr – Meyer) constructed an ideal, for each positive integer r, I_{X_r} generated by $22r-3$ quadrics ~~and~~ ~~R~~ in $22r-1$ variables. But $\operatorname{Reg}(I_{X_r}) \geq 2^{2^r}$

Anyway, one is expecting doubly exponential behavior for the regularity for arbitrary ideals.

The problem is that I_{X_r} is far frome being a prime ideal [MP]'s idea is to construe a prime ideal in a new polynomial ring with regularity comparable to S/I_{X_r}.

Start with I (or I_{X_r}). ~~a~~ The classical way ~~to~~ to an integral domain is to consider $\mathrm{Rees}(I) = S[It] \subseteq S[t]$. But ~~it~~ it is very difficult to control the syzygies of $\mathrm{Rees}(I)$. Instead

[MP] Consider

$R = S[I\bar{I}, t^2] \subseteq S[t]$.

We'll try to control the Betti numbers of R.

We start with a minimal resolution of S/I

$\mathcal{F}_\bullet$: $\quad \cdots\ F_2 \xrightarrow{\partial_0} F_1 \xrightarrow{\partial_1} F_0 \to S/I \to 0$,

$F_0 = S$. $\quad F_1 = \bigoplus_{i=1}^{m} S(-d_i)$

I is minimally generated by f_i where $\deg f_i = d_i$

$F_2 = \bigoplus_{j=1}^{N} S(-b_j)$

∂_0 is a $m \times N$ matrix (c_{ij})

$$(f_1 \cdots f_m)\begin{pmatrix} c_{ij} \end{pmatrix} = (\overbrace{0, \cdots 0}^{N})$$

$c_{ij} \in S_{b_j - d_i}$.

For future we also consider

$\mathcal{Y}_1 \quad \cdots\ F_2 \to F_1 \to \ker \partial_0 \to 0$.

Set $T = S[w_1, \ldots w_m\ u]$

where $\deg w_i = d_i + 1$ $\deg u = 0$

We've a S-algebra surjection ψ from T to R

by sending $w_i \longmapsto f_i t$ and u to t

Set $Q = \ker \psi$ is a prime ideal.

Consider the following two sets

$A = \{ \cancel{f_i f_j}\ w_i w_j - u f_i f_j \mid 1 \leq i \leq j \leq m \}$

$B = \{ \sum_{i=1}^{m} c_{ij} w_i \mid 1 \leq j \leq N \}$.

Proposition 4.7 $A \cup B$ is a minimal set of generators of Q, ~~as~~ Q is a prime id and $u \notin Q$.

For $b/\mathfrak{a}\wedge b$

$$W_\ell\left(\sum_{i=1}^{m} c_{ij} W_i\right) \in \langle W_{ij} \mid 1 \le i \le j \le m\rangle$$

So b is a S-module

$$\hookrightarrow \ker \tau \quad \bigoplus_{1}^{N} S(-b_j-1) \xrightarrow{\tau} b/\mathfrak{a}\wedge b$$

Lemma 4.8. $\ker \tau = (\ker \partial_1)(-1)$.

So $b/\mathfrak{a}\wedge b$ is resolve as a S-module

by $\mathcal{Y}_1(-1) \longrightarrow b/\mathfrak{a}\wedge b \to 0$.

Regard T as $S \otimes_k k[W_1 \cdots W_m]$

We see $b/\mathfrak{a}\wedge b$ is resolved by

$$\mathcal{Y}_1(-1) \otimes \mathrm{Kos}(W_1 \cdots W_m) = B.$$

Lemma 4.9

Using a "mapping cone" construction one sees the $\bar{T}/G$ has a minimal resolution of the form $D_\cdot$ ~~$\mathcal{F}(-1) \otimes \mathrm{Kos}(w_1 \ldots w_m)^{[-1]}$~~ $\oplus \mathcal{G}$ $\mathcal{F}$:

when $D_q = \mathcal{G}_q \oplus B_{q-1}$. $\square$

Since $\mathcal{F}_1$ has a high regularity

$B = \mathcal{F}_1 \otimes \mathrm{Kos}(w_1 \ldots w_m)$ has so high ~~regula~~ regularity.

We conclude $\bar{T}/G$ has high regular.

and T/G has high regularity. and G is a homogen prime ideal. ~~Lemma~~

The final ~~final~~ difficulty is that

$T = \not\! S[w_1 \ldots w_m\ u]$ when

$\deg w_i = d_i + 1$ and $\deg u = 2$. So

T is not ~~a~~ ~~ster~~ not a standard graded polynomial ideal.

Repeat this process, we finally

arrive a polynomial ring

$\tilde{T} = \cancel{k[} S[y_1, z_1 \cdots y_m, z_m\, y_{m+1}\, z_{m+1}']$

an a prime $\tilde{G}$ in $\tilde{T}$ such

$\tilde{T}/\tilde{G}$ have same graded Betti numbers

as T/G. So it have very high

regularity.

It remain to find $\deg \tilde{T}/\tilde{G}$.

Lemma. 4.10 $\deg \tilde{T}/\tilde{G} = 2 \prod_{i=1}^{m} (d_i + 1)$.

In koh example $d_i = 2$

So $\deg \tilde{T}/\tilde{G} = 2 \cdot 3^m = 2^{22r-2}$ $m = 22r - 3$.

$\cancel{\mathrm{Reg}\, S/I}$

Ex. 4.11 $\mathrm{Reg}(\tilde{T}/\tilde{G}) = \mathrm{Reg}(T/G) = \mathrm{Reg}(\bar{T}/\bar{G})$

$\mathrm{Max}\{\mathrm{Reg}\, \bar{y}_1 + \mathrm{reg}\, K_{\bar{\omega}}\bar{T}, \mathrm{Reg}\, G\}$

$= \mathrm{Reg}(S/I) + 2 + \sum_{1}^{m} d_i \geq 2^{2^r}$

u is not a zero divisor of T/G.

Set $\bar{\bar{T}} = T/M$ and $T/\langle G, w\rangle = \bar{T}/\bar{G}$.

T/G and $\bar{\bar{T}}/\bar{G}$ have the same Betti numbers. A generate an ideal $\mathfrak{a}$ in $\bar{\bar{T}}$ and B generate an ideal $\mathfrak{b}$ in $\bar{\bar{T}}$. There is an exact sequence

$$0 \to \frac{\mathfrak{a}+\mathfrak{b}}{\mathfrak{a}} \to \bar{\bar{T}}/\mathfrak{a} \to \bar{\bar{T}}/\mathfrak{a}+\mathfrak{b} = \bar{T}/\bar{G} \to 0$$

$$\frac{\mathfrak{a}+\mathfrak{b}}{\mathfrak{a}} \cong \mathfrak{b}/\mathfrak{a}\mathfrak{b}.$$

$\mathfrak{a} = \langle w_i w_j \mid 1 \le i \le j \}$

$\bar{\bar{T}}/\mathfrak{a} = S \otimes_k k[w_1 \ldots w_m]/M_w^2$.

We know $k[w_1 \ldots w_m]/M^3$ is resolved by only an Eagon Northcott complex. $\mathcal{G}'$

$\bar{\bar{T}}/\mathfrak{a}$ is resolved by $\mathcal{G} = S \otimes_k \mathcal{G}'$.

For $b/\mathfrak{a}\cap b$

$$W_l\left(\sum_{i=1}^{m} c_{ij} W_i\right) \in \langle W_{ij} \mid 1 \le i \le j \le m\rangle$$

So b is a S-mod

$$\hookrightarrow \ker \tau \quad \bigoplus_{1}^{N} S(-b_j-1) \xrightarrow{\tau} b/\mathfrak{a}\cap b$$

Lemma 4.8. $\ker \tau = (\ker \otimes \partial_1)(-1)$.

So $b/\mathfrak{a}\cap b$ is resolve as a S-mod

by $\mathcal{F}_1(-1) \longrightarrow b/\mathfrak{a}\cap b \to 0$.

Regm T as $S \otimes_k k[W_1 \cdots W_m]$

We see $b/\mathfrak{a}\cap b$ is resolv by

$$\mathcal{F}_1(-1) \otimes K_\infty(W_1 \cdots W_m) = B.$$

Lemma 4.9

Using a "mapping cone" construction one sees the $\overline{T}/G$ has a minimal resolution of the form $D_\cdot$ ~~$\mathcal{F}(-1) \otimes \mathrm{Kos}(w_1 \ldots w_m)$~~$[-1] \oplus \mathcal{G}$ $\mathcal{G}$: when $D_q = \mathcal{G}_q \oplus B_{q-1}$. $\square$

Since $\mathcal{F}_1$ has a high regularity $B = \mathcal{F}_1 \otimes \mathrm{Kos}(w_1 \ldots w_n)$ has so high ~~regula~~ regularity.

We conclude $\overline{T}/G$ has high regularity and I/G has high regularity. and G is a homogen prime ideal. ~~Lemma~~

The final ~~final~~ difficulty is that $T = \oplus S[w_1 \ldots w_m\ u]$ where $\deg w_n = d_n + 1$ and $\deg u = 2$. So T is not ~~a~~ ~~ster~~ not a standard graded polynomial ideal.

Repeat this process, we finaly

arrive a polynomial ring

$\widetilde{T} = \cancel{k} S[y_1, z_1 \cdots y_m, z_m\ y_{m+1}\ z_{m+1}]$

an a prime $\widehat{G}$ in $\widetilde{T}$ such

$\widetilde{T}/\widehat{G}$ have same graded Betti numbers

as T/G. So it have very high

regularity.

It remain to find $\deg \widetilde{T}/\widehat{G}$.

Lemma. 4.10 $\deg \widetilde{T}/\widehat{G} = 2 \prod_{i=1}^{m} (d_i+1)$.

In koh example $d_i = 2$

So $\deg \widetilde{T}/\widehat{G} = 2 \cdot 3^m = 2^{22r-2}$ $m = 22r-3$.

$\cancel{Reg\ S/I}$

Ex. 4.11 $Reg(\widetilde{T}/\widehat{G}) = Reg(T/G) = Reg(T/\overline{G})$

$Max\{Reg\ y_1 + reg\ k_{\text{co}}\ \overline{T},\ Reg\ G\}$

$= Reg(S/I + 2 + \sum_1^m d_i \geq 2^{2^r}$

Example ~~4.6~~ C is a nonhyperelliptic curve of genus g $(g \geq 3)$. Then K_C is very ample

$$C \xhookrightarrow{|K_C|} \mathbb{P}(H^0(K_C)) = \mathbb{P}^{g-1}$$

$$0 \to I_C \to \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_C \to 0$$

$$H^1(\mathcal{O}_C(1)) = H^1(\omega_C) \neq 0 \qquad j \geq 2$$

$$H^1(\mathcal{O}_C(j)) = H^1(\omega_C^j) \cong H^2(\mathcal{I}_C(j)) = 0$$

So ~~I~~ One checks that ~~I~~ $\mathcal{I}_C$ is 4-regular. The fact that $H^1(\mathcal{I}_C(j)) = 0$ for all j is ~~Gul~~ a theorem of Noether. We'll describe that next time. So the height of Betti table is 4.

$E_{\mathrm{Kos}(w_1 \cdots w_n)}$

$\mathrm{Kos}(w_1 \cdots w_n)$ resoln $S/M = k$ $\deg w_i = d_i + 1$

$$H_{S/M} = 1 = \frac{\cancel{E_{\mathrm{Kos}(w}}\ E_{\mathrm{Kos}(w_1 \cdots w_n)}}{\prod (1 - t^{d_i+1})}$$

$$\cancel{H}\ E_{\mathrm{Kos}(w_1 \cdots w_n)} = \prod (1 - t^{d_i+1}).$$

If G is the Eagon complex of S/M^2.

$$H_{S/I^2} = 1 + \sum_1^m t^{d_i+1}$$

$$\text{So } E_{S/I^2} = \left(\prod_{i=1}^m (1 - t^{d_i+1})\right)\left(1 + \sum_1^m t^{d_i+1}\right).$$

$$E_{Y_1} = E_Y - (1 + \sum t^{d_i}).$$

$$E_{Y_1(-1)[1]} = (-1)\, t \cdot E_{Y_1}.$$

By choosing $r \gg c$, we find counter-example to the Eisenbud-Goto regularity conjecture.

Hints for Exercise 4.11

For each ~~finite~~ ~~general~~ a ideal I ~~S-m S-module~~ M

$S = k[x_1 \cdots x_r]$ $\quad \deg x_i = d_i$

The Hilbert series ~~$H_M(x) = \sum \dim M_m$~~

$$H_{S/I}(x) = \frac{E_{S/I}(x)}{\prod (1-t^{d_i})} = \cancel{\sum_{i,j} (-1)^i}$$

$E_{S/I}(x)$ can be computed by E_F.

where F_i is a resolution of S/I.

$$F_i = \oplus S(-j)^{\beta_{ij}}$$

Then $E_{S/I}(x) = \sum (-1)^i \beta_{ij} t^j$, $= E_F$.

When all $d_i = 1$, ~~do~~ Write $E_{S/I}(x) = (1-t)^c \cdot g(x)$, where $g(1) \neq 0$

& Then $c = \operatorname{codim} S/I$ and $g(1) = \deg S/I$.