2020 AG Summer School

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July 2020

References for Lecture I-III: Hartshorne "Algebraic Geometry" Chapter I and Chapter II; Vakil "Foundations of Algebraic Geometry" Part II and Part V

1 Projective Scheme

All the rings will be commutative.

1.1 A warm up for projective geometry

* **Projective spaces as complex manifolds**

  $\mathbb{P}^n_{\mathbb{C}} := \mathbb{C}^n - \{(0,\dots,0)\}/\sim$ has a complex manifold structure by associating a holomorphic local charts.

  Definition 1.1.1 (projective complex manifold)**.** A complex manifold M is said to be projective if there is a closed embedding $M \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$ for some n.

  Typical examples (from last week)

  - The compact Riemman surfaces of genus g
  - The Grassmannian $\mathrm{Gr}(r,n)$ can be embedded into $\mathbb{P}^{\binom{n}{k}-1}$ via the Plücker embedding.
  - product, $\mathbb{P}^n$-bundle, polarized families over projective objects

* **Projective spaces as schmems/varieties**

  We have seen from last week: $\mathbb{P}^n_{\mathbb{C}}$ can be viewed as a scheme obtained by gluing $n+1$ open subsets

$$U_i \cong \mathbb{A}^n_{\mathbb{C}} = \mathrm{Spec}\left(\mathbb{C}\left[\frac{X_0}{X_i},\dots,\frac{X_n}{X_i}\right]\right)$$

  along the overlaps $U_{ij} = U_i \cap U_j$ via the transistion function $\frac{X_j}{X_i} \mapsto \frac{X_i}{X_j}$.

  The projective schemes over $\mathbb{C}$ are closed subschemes of $\mathbb{P}^n_{\mathbb{C}}$ under Zariski topology. The projective complex varieties are obtained by taking the closed points of projective schemes. Such varieties can be viewed as the solution set of homogenous polynomial equations

$$f_1(x_0,\dots,x_n) = \dots = f_m(x_0,\dots,x_n) = 0.$$

where f_i are homogenous.

- **Chow's Theorem/GAGA by Serre**: there is an equivalence

$$\{\texttt{projective complex manifolds }\} \Leftrightarrow \{\texttt{Projective complex varieites}\}$$

Goal of today: *functorial algebraic constructions of projective objects*

1.2 Proj constructions

The Spectrum functor defines an equivalence

$$\text{Spec} : \{\texttt{rings}\} \longrightarrow \{\texttt{affine schemes}\}$$
$$R \mapsto \text{Spec } R$$

The projective schemes can be obtained via so called Proj functor.

Definition 1.2.1 (**Proj** construction for graded ring)**.**
Let $S = \bigoplus_{d\geq 0} S_d$ be a graded ring and $S_+ = \bigoplus_{d>0} S_d$ the irrelevant ideal. Then

$$\text{Proj } S = \{\mathfrak{p} \in \text{Spec } S |\ \mathfrak{p} \text{ is homogenous}, S_+ \not\subset \mathfrak{p}\}$$

We endow it with the induced topology.

- $\forall f \in S$ homogenous of degree d, there is a standard open subset

  $$D_+(f) = \{\mathfrak{p} \in \text{Proj } S | f \notin \mathfrak{p}\} \cong \text{Spec } S_{(f)}$$

  where $S_{(f)}$ is the subring of S_f consisting of elements of the form r/f^n with r homogeneous and $\deg(r) = nd$.

- The structure sheaf $\mathcal{O}_{\text{Proj } S}$ on Proj S is the unique sheaf of rings $\mathcal{O}_{\text{Proj } S}$ which agrees with $\mathcal{O}_{\text{Spec } (S_{(f)}}$ on the standard open subset $D_+(f)$.

Example 1. 1. When $S = k[x_0,\dots,x_n] = \bigoplus S_d$ with the usual grading, then Proj $S = \mathbb{P}^n_k$.

2. Write $T = k[y_0,\dots,y_m] = \bigoplus T_d$. Then

$$\text{Proj } (\bigoplus_d S_d \otimes T_d) = \mathbb{P}^n_k \times \mathbb{P}^m_k.$$

FACT: Proj defines a functor

$$\text{Proj} : \{\texttt{graded ring over A}\} \to \{\texttt{projective scheme over A}\}$$

Some examples for morphisms between projective varieties.

Example 2.

(1) Veronese or d-uple embedding: $\varphi_d : \mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{d}-1}$ sending $[x_0, \ldots, x_n]$ to $[x_0^d, x_0^{d-1}x_1, \ldots, x_n^d]$.

(2) Segre embedding: $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{nm+n+m}$ sending $([x_0, \ldots, x_n], [y_0, \ldots, y_m])$ to $[x_0y_0, \ldots, x_ny_m]$.

Note that (2) also implies that the product of projective varieties over k remains projective.

The construction of Proj of a graded sheaf gives rise to a projective morphism.

Definition 1.2.2 (Proj construction of graded sheaf)**.**
• A graded quasicoherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules means

$$\mathcal{F} = \bigoplus_{d \geq 0} \mathcal{F}_d$$

satisfying $\mathcal{F}_d \cdot \mathcal{F}_{d'} \subseteq \mathcal{F}_{d+d'}$ and $\mathcal{F}_0 = \mathcal{O}_X$.
• We can define Proj $\mathcal{F}$ by gluing the scheme Proj $\mathcal{F}(U)$, $U \subseteq X$.

Example 3. If $\mathcal{E}$ is a locally free sheaf on X, then $\mathrm{Sym}^\bullet \mathcal{E} = \bigoplus \mathrm{Sym}^d \mathcal{E}$ is a graded $\mathcal{O}_X$-module. We obtain a projective bundle

$$\mathbb{P}(\mathcal{E}) = \mathrm{Proj}\ (\mathrm{Sym}^\bullet \mathcal{E})$$

over X.

Basic properties of a projective scheme.

(a) Let X be a projective variety over k. Then X is proper and $H^0(X, \mathcal{O}_X) = k$.

The converse is almost true:

Chow's Lemma: *Every proper variety is birational to a projective variety.*

(b) (Twisted sheaf) Suppose $S = k[x_0, \ldots, x_n]$ is generated by S_1. The projective scheme Proj S carries a natural invertible sheaf $\mathcal{O}_S(1) := \widetilde{S}(1)$.

E.g. the projective space $\mathbb{P}^n_k$ carries a natural invertible sheaf $\mathcal{O}_{\mathbb{P}^n_k}(1)$. Hence the projective subvariety $X \subseteq \mathbb{P}^n_k$ can be endowed with an invertible sheaf $\mathcal{O}_X(1)$ via restriction.

2 Geometry of projective varieties

Classical problem: find ♯ of polynomial equations. Geometrically, this is related to how varieties intersects.

The answer of this problem is to relate ♯ to some invariants of projective varieties.

2.1 Invariants of projective varieties

A motivating example is

Example 4 (Gauss' fundamental theorem of algebra)**.** The polynomial equation $f(z) = 0$ has $\deg(f)$ solutions (with multiplicity) in $\mathbb{C}$. Equivalently, the homogenous polynomial equation $f(x, y) = 0$ has $\deg(f)$ solutions.

For three variables, the fundamental result is the following:

Theorem 2.1.1 (Bézout theorem for plane curves)**.** *Let f, g be two distinct irreducible homogenous polynomials in $k[x, y, z]$. The equations*

$$f(x, y, z) = g(x, y, z) = 0$$

have $\deg(f) \cdot \deg(g)$ *solutions (with multiplicity).*

In other words, the two plane curves $C_1 = \{f(x, y, z) = 0\}$ and $C_2 = \{g(x, y, z) = 0\}$ in $\mathbb{P}^2$ meet at $\deg(f)\deg(g)$ points.

Remark. The Bézout's theorem tells that any two closed curves in $\mathbb{P}^2$ will have non-empty intersections. Note this fails for affine varieties, i.e. two affine lines in $\mathbb{A}^2$ do not necessarily meet.

The higher dimensional generalization requires the concept of Hilbert polynomial.

∗ Hilbert polynomial of projective varieties

Definition 2.1.2. Let $\mathcal{F}$ be a coherent sheaf on a projective scheme $X \subseteq \mathbb{P}^n$. By **Hilbert-Serre**, there eixsts a polynomial $P_{\mathcal{F}}(z) \in \mathbb{Q}[z]$ such that

$$P_{\mathcal{F}}(d) = \chi(\mathcal{F}(d)) = \sum_{i \geq 0} (-1)^i h^i(X, \mathcal{F}(d))$$

for $d >> 1$, where $\mathcal{F}(d) = \mathcal{F} \otimes \mathcal{O}_X(d)$. $P_{\mathcal{F}}(z)$ is the Hilbert polynomial of $\mathcal{F}$ and $P_X := P_{\mathcal{O}_X}$ is called the Hilbert polynomial of X in $\mathbb{P}^n$.

Facts for P_X (not very trivial)

1. $P_X(d) = h^0(X, \mathcal{O}_X(d))$ for d sufficiently large due to the Serre vanishing theorem, i.e. $H^i(X, \mathcal{F}(d)) = 0$ for $i > 0$ if $\mathcal{F}$ is coherent and d sufficiently large.
2. First invariant: $\deg(P_X) = \dim X = m$.

3. Seconding invariant: leading coefficients of P_X is $\frac{\deg(X)}{m!}$.
4. Invariant from the constant term: the arithmetic genus of X: $(-1)^m(P_X(0)-1)$.

All these invariants are deformation invariant.

Example 5 (Invariants determines the geometry)**.** 1. if $\deg(X) = 1$, then X is a projective linear subspace in $\mathbb{P}^n$.

2. More generally, if X is non-degenerate in $\mathbb{P}^n$, then $\dim X + \deg(X) \geq n+1$.

$*$ Bézout's theorem

With the knowledge of the degree, we can state Bézout's theorem in arbitrary dimensional projective space.

Theorem 2.1.3. *Let X be a projective variety in $\mathbb{P}^n_k$ with $\dim X \geq 1$ and H be a hypersurface not containing X. Denote by Z_i the irreducible components of the intersection of H and X. Then*

$$\deg(X) \cdot \deg(H) = \sum_{Z_i} \mu(X, H; Z_i) \deg(Z_i)$$

where $\mu(X, H; Z_i)$ is the intersection multiplicity at Z_i.

The proof relies on computing the Hilbert polynomials via the short exact sequence

$$0 \to \mathcal{O}_X(-\deg(H)) \to \mathcal{O}_X \to \mathcal{O}_{X \cap H} \to 0$$

Remark: if $\mathfrak{p}_i$ is the prime ideal corresponds to Z_i, then $\mu(X, H; Z_i)$ is the length of $(k[x_0, \dots, x_n]/(I_X + I_H))_{\mathfrak{p}_i}$ as a $k[x_0, \dots, x_n]_{\mathfrak{p}_i}$-module.

Important consequences

- For projective variety X in $\mathbb{P}^n$ of dimension d, the intersection number with d general hyperplanes

$$H_1 \cdot H_2 \dots \cdot H_d \cdot X$$

  is positive. As all hyperplanes are linearly equivalent, it is the same as $H^d \cdot X > 0$.
- More generally, if we call the intersection $L := X \cap H$ the hyperplane class on Y, then $L^d \cdot Y > 0$ for any subvariety $Y \subseteq X$ of dimension d.

2.2 Generic intersection

Among the questions for intersection multiplicity, a natural one is when the intersection multiplicity will be one.

Definition 2.2.1. Let X be a variety over k. A point $p \in X$ is smooth $\dim X = \dim T_p X$.

Theorem 2.2.2 (Bertini Theorem)**.** *Let $X \subseteq \mathbb{P}(V) \cong \mathbb{P}^n$ be a smooth subvariety of dimension greater than zero. Then for a generic hypersurface H, $Y = X \cap H$ is again smooth.*

Proof. 1. Note that the set of hyperplanes is parametrized by the dual projective space $\mathbb{P}(V^\vee)$.

2. To say that a hyperplane is generic is equivalent to saying that there is a nonempty open subset $U \subseteq \mathbb{P}(V^\vee)$ consisting of points corresponding to that hyperplane and such that each hyperplane in U possesses the desired property.

3. $H \cap X$ will be smooth at x if $T_x X \not\subset T_x H$.

4. Consider the subset

$$Z = \{(H, x) \mid x \in H, T_x X \subset T_x H\} \subseteq \mathbb{P}(V^\vee) \times X,$$

it is a closed subset.

5. The set of H in $\mathbb{P}(V^\vee)$ for which $H \cap X$ is singular is the image of Z via the projection $\mathbb{P}(V^\vee) \times X \to \mathbb{P}(V^\vee)$.

6. The assertion follows by an easy dimension count: $dim(Z) = n - 1$. □

A more general statement is as follows:

Theorem 2.2.3. *Suppose* $\mathrm{char}(k) = 0$*. Then for any linear system $f : X \dashrightarrow \mathbb{P}^n_k$ and H a generic hyperplane, the pullback $f^{-1}(H)$ is smooth outside the base locus of f.*

It fails in positive characteristic fields because the existence of purely inseparable map.

3 Ampleness criteria

Guiding Problem: How to determine the projectivity of a scheme?

Answer: projectivity $\Longleftrightarrow$ existence of ample line bundle.

3.1 Linear system

We may identify three concepts: line bundle, Cartier divisor and invertible sheaf.

- Given a morphism $\varphi : X \to \mathbb{P}^n$, we obtain an invertible sheaf $\varphi^*(\mathcal{O}_{\mathbb{P}^n}(1))$.
- Conversely, given a base point free invertible sheaf $\mathcal{L}$, we obtain morphisms

$$\varphi_{\mathcal{L}} : X \to |\mathcal{L}| = \mathbb{P}(H^0(X, \mathcal{L})^\vee)$$

So X is projective if and only if there exists a base point free invertible sheaf $\mathcal{L}$ such that φ_L is a closed embedding.

Lemma 3.2. *φ_L is a closed embedding if and only if the sections of $\mathcal{L}$ separate points and tangent vectors, i.e.*

1. *$\forall p, q \in X$, there exists $s \in H^0(X, \mathcal{L})$ such that $s(p) = 0$ and $s(q) \neq 0$.*
2. *$\forall p \in X$, the image of $s \in H^0(X, \mathcal{L})$ with $s(p) = 0$ spans $(T_p X)^\vee$.*

Definition 3.2.1. We say that a Cartier divisor D (or an invertible sheaf) is very ample if $\varphi : X \to \mathbb{P}^n$ defines an embedding of X. We say that D is ample if mD is very ample for some $m \in \mathbb{N}$.

Useful Properties

- (very) Ample + (very)Ample is (very) ample. (Segre map)
- (very) Ample + Base point free is (very) ample. (Segre map)
- Inveritible + sufficiently ample is ample (above+Serre's theorem)
  A consequence is the every divisor is the difference of two ample divisors.
- Ampleness $\Leftrightarrow$ for $f \in H^0(X, \mathcal{L}^{\otimes n})$, the open susbets $X_f = \{x \in X | \, f(x) \neq 0\}$ which are affine form a base of the topology of X. (local check)

The guiding problem becomes how to characterize (very) ample line bundles/divisors.

Proposition 3.2.2 (Serre's cohomological criterion)**.** *TFAE*

- *D is ample*
- *for any quasi-coherent sheaf $\mathcal{F}$, $H^i(X, \mathcal{F}(mD)) = 0$ for all $i > 0$ and m sufficiently large.*

As a corollary, if $f : X \to Y$ is a finite morphism and D a Cartier divisor on Y, then $f^*(D)$ is ample if D is ample.

Proof. We only prove the corollary. If f is finite, then

$$H^i(X, \mathcal{F}(mf^*D)) = H^i(Y, f_*\mathcal{F}(mD)) = 0$$

for $m >> 0$. □

Note that this fails if f is no longer finite.

3.3 Ampleness v.s. positivity

In complex geometry, a famous result is

Kodaira embedding: Let X be a compact complex manifold. A line bundle L is positive (i.e. it admits a Hermitian metric whose curvature form is positive (1,1)-form) iff L is ample.

Example 6. Any positive degree divisor D on a curve is ample.

Nakai-Moishezon's criterion shows that ampleness is in fact numerical properties.

Theorem 3.3.1 (Nakai-Moishezon)**.** *If D is a Cartier divisor on a projective k-scheme X, and for every subvariety Y of X of dimension n, $(D^n \cdot Y) > 0$, then D is ample.*

Proof. Sketch of the proof. The proof proceeds by induction on the dimension of X. When $\dim X = 1$, this is from last week. Suppose it holds for $\dim X \leq n-1$.

Step 1. for $m >> 0$, $|mD|$ is non empty.

Write $D = A - B$ as a difference of two very ample divisors A, B. Then we have

$$0 \to \mathcal{O}_X(mD - B) \to \mathcal{O}_X((m+1)D) \to \mathcal{O}_A((m+1)D) \to 0$$

and

$$0 \to \mathcal{O}_X(mD - B) \to \mathcal{O}_X(mD) \to \mathcal{O}_B((mD) \to 0.$$

By long exact sequence, inductive hypothesis and Serre's vanishing $H^i(\mathcal{O}_A((m+1)D)) = H^i(\mathcal{O}_B(mD)) = 0$ for $i > 0$. One obtain

$$H^i(X, \mathcal{O}_X(mD)) = H^i(X, \mathcal{O}_X((m+1)D))$$

for $i \geq 2$ and m large enough.

By RR, we know

$$\chi(X, mD) = h^0(mD) - h^1(mD) + constant$$

is asymptoically of the form $\frac{m^n D^n}{n!} + \ldots$, which increases to $+\infty$ as $m \to \infty$. Thus $h^0(mD)$ is nonzero.

Step 2. for $m >> 0$, mD is base point free. Similar idea as above.

Step 3. Consider the morphism $\varphi : X \to |mD|$, then $D = \varphi^* H$. If φ is not finite, then φ contracts some curve C. But

$$D \cdot C = \phi^* H \cdot C = H \cdot \phi_* C = 0$$

which is a contradiction.

□

For X being a surface, this criterion is equivalent to the following conditions

- $L \cdot C > 0$ for any curve $C \subseteq X$.
- $L^2 > 0$

3.4 Cones and Kleiman's ampleness criteria

Let X be a scheme over k. Note that the sum of ample divisors remain ample, this yields the concept of ample cones:

Definition 3.4.1. $\mathrm{Amp}(X)$: the cone generated by ample divisors.

A consequence of Nakai's criterion is that $\mathrm{Amp}(X)$ is open. Moreover, we have a complete description of this cone only using the intersection with curves.

Definition 3.4.2. A divisor D on X is nef if $D \cdot C$ for any curve $C \subseteq X$. Then we define $\mathrm{NE}(X)$ as the cone generated by nef divisors.

Theorem 3.4.3 (Kleiman)**.** *Suppose X is a projective scheme over k. Then*

1. *The nef cone is the closure of the ample cone.*
2. *The ample cone is the interior of the nef cone.*

Example 7. The ample cone of $\mathbb{P}^1 \times \mathbb{P}^1$ is

$$\{aD_1 + bD_2, a > 0, b > 0\}$$

where D_1 and D_2 are two rulings.

Example 8. The ample cone of $\mathrm{BL}_p\mathbb{P}^2$ is

$$\{aL - bE, a > b > 0\}$$

where $L = \pi^* \mathcal{O}_{\mathbb{P}^2}(1)$ and E is the exceptional divisor. In particular, $2L - E$ is ample (in fact very ample).

Similar computation shows that for $\mathrm{BL}_p\mathbb{P}^3$, the divisor $2L - E$ is ample.