Lecture V: Hilbert Scheme

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Goal: find the parametrization space of projective schemes in a fixed projective space.

Locally free sheaf $\rightsquigarrow$ vector bundle.

We will form the question as a moduli problem.

free sheaf $\rightsquigarrow \mathcal{O}_S^{\oplus n} \rightsquigarrow S \times V$

Two functors

(I). The functor **G(r,V)** : *Scheme* —> *Set* is defined by

$\dim_k V = n$. vector space.

G(r,V)(S)={ rank r sub-bundle of S ×V }/~

$G(r, n)$

In particular, **G(r,V)**(pt)={ r-dim linear subspaces of V }=: Grass (r, V).

(II). Let p(z) be a rational polynomial. The functor **Hilb(p, V)** : *Scheme* —> Set is defined by

$\mathbb{P}^n_k$ $\quad \mathbb{P}^n_S$.

Hilb(p, V)(S)={ $X \xrightarrow{\pi} S$ is flat family of closed subschems in $\mathbb{P}(\mathbb{V})$ with Hilbert polynomial p(z)}/~

$\pi^{-1}(s) \subseteq \mathbb{P}^n_k$.

closed subscheme with Hil. polynomial p(z)

Representability

smooth

Theorem A. *The functor* $G(r,V)$ *is represented by the projective scheme Grass(r,V) of dimension r(n-r), where n=dim V.*

E.g. if r=1 or dim V-1, Grass(r,V)= $\mathbb{P}(\mathbb{V})$ is the projective space.

Theorem B. *The Hilbert functor* $Hilb(p, V)$ *is representable by a projective scheme, called the Hilbert scheme of closed subschemes in* $\mathbb{P}(\mathbb{V})$ *with respect to p.*

Remark.

projective

Thm A is a special case of Thm B, as $G(r,V)$ parametrizes the r-1 dimensional linear space in $\mathbb{P}(\mathbb{V})$

families of $\mathbb{P}^{r-1} \subseteq \mathbb{P}^n$

$Hilb(p, V)$

p = Hil poly.

of $\mathbb{P}^{r-1} \subseteq \mathbb{P}^n$

Remark:

- There are several proof of Thm A. $>$ Thm B
- Thm B follows from Thm A.
- Reference: ≪ Rational curves on algebraic varieties ≫ J. Kollár Chapter I.

  (other pfs.)

- Proof of Thm A.

  (local argument)

  Idea: I will find some $\xrightarrow{\text{open}}$ subfunctors of $G(r, V)$.

  and show such functors are representable by $\mathbb{A}^N$

Then I can show these functors can glue.

$\Rightarrow$ $G(r,V)$ is representable by the scheme glued by those $\mathbb{A}^N$.

e.g. $r=1$, $G(r,V) = \mathbb{P}(V) \cong \mathbb{P}^n$.

glued by $(n+1)$ $\mathbb{A}^n$.

· 1). modify our functor.

$\forall S$, $G(r,V)(S) = \{$ sub-bundles of rank r of $S \times V \}$

$\Longleftrightarrow \{ \mathcal{O}_S^{\oplus n} \twoheadrightarrow Q \}$

locally free sheaf of rank $n-r$.

$\forall I = \{\hat{j}_1, \hat{j}_2, \cdots \hat{j}_{n-r}\} \subseteq \{1, 2, \cdots n\}$. $|I| = n-r$.

<u>Def:</u> $G_I \subseteq G(r, V)$. subfunctor.

$$G_I(S) = \{ \mathcal{O}_S^{\oplus n} \twoheadrightarrow Q \mid \underbrace{\bigoplus_{i=1}^{n-r} \mathcal{O}_S\langle \hat{j}_i \rangle \to \mathcal{O}_S^{\oplus n} \twoheadrightarrow Q}_{\twoheadrightarrow} \}$$

remains surjective.

$$\cong \{ \underline{\mathcal{O}_S^{\oplus n} \twoheadrightarrow \bigoplus_{i=1}^{n-r} \mathcal{O}_S \langle \hat{j}_i \rangle} \}$$

To determine $\underline{q: \mathcal{O}_S^{\oplus n} \twoheadrightarrow \bigoplus_{i=1}^{n-r} \mathcal{O}_S \langle \hat{j}_i \rangle}$. <u>matrix</u>.

is determined by maps between global sections.

$$G_I(S) \cong \prod_{j=1}^{r} \Gamma(S, \mathcal{O}_S^{\oplus n-r}).$$ $\boxed{(Ex)}$

So G_I is nothing but taking global sections

$\downarrow$

G_I is represented by $\mathbb{A}^1 \times \mathbb{A}^1 \times \mathbb{A}^1 \cdots \times \mathbb{A}^1 = \mathbb{A}^N$.

$$G_I(pt) = \{ E \subseteq V,\ \dim E = r \mid E \cap F = 0 \} \subseteq \operatorname{Grass}(r, V).$$

open subset.

$$\cap$$

$$G(r, V) = \{ E \subseteq V,\ \dim E = r \}.$$

I

$\downarrow$

$F \subseteq V$

$\dim F = n - r$.

e.g. $r = 1$, $G_I(pt) = \{ x_i \neq 0 \} \subseteq \mathbb{P}^n$

$$\wr\!\wr$$

$$\mathbb{A}^n.$$

- **Remark:** $\mathrm{Grass}(r, V) \hookrightarrow \mathbb{P}(\wedge^r V)$

Plücker embedding.

$E \subseteq V \mapsto \wedge^r E \in \mathbb{P}(\wedge^r V)$

- Proof of Thm B.

$\mathrm{Hilb}(p, V)(S) = \{ \text{family over } S \}$

- Take $S = pt$, How to describe $\mathrm{Hilb}(p, V)(pt)$ as a scheme.

$X \in \mathrm{Hilb}(p, V)(pt), \iff X \subseteq \mathbb{P}(V)$

$P_X = p$

- X is deterined by $\underline{\mathcal{I}_X}$: ideal sheaf of X in $\mathbb{P}(V)$.

- FACT: We can chose $N >> 0$, s.t.

$$\mathcal{I}_X(N) := \mathcal{I}_X \otimes (\mathcal{O}_{\mathbb{P}(V)}(1))^{\otimes N}.$$

is globally generated.

(prof: see the prof of ampleness criteria).

$$\Rightarrow \underbrace{H^0(\mathbb{P}(V), \mathcal{I}_X(N))}_{\text{determines } X} \subseteq H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(N)).$$

$$\begin{array}{c} X \\ \{\text{subschemes of } \mathbb{P}(V) \text{ with } P_X = p\} \end{array} \quad \underline{\text{take } N \gg 0.}$$

$$\downarrow$$

$$\mathrm{Grass}(p(N),\ H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(N)))$$

$$X \longmapsto H^0(X, \mathcal{O}_X(N)) \text{ of dim } p(N)$$

- FACT: if $N \gg 0$, this map is a closed embedding
- In other words, $\mathrm{Hilb}(p, V)(pt) \subseteq \mathrm{Grass}(r, N)$ <u>for some r, N.</u>

· <u>Examples</u>.

(1) Hilbert scheme of pts. (Last week)

Hilbert scheme of n-pts on $\mathbb{P}^1 \cong \mathbb{P}^n$.

$\wr$

n-pt on C. $\cong J^n(C)$

$$\downarrow$$

$$J(C)$$

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(2) Hilbert scheme of curves in $\mathbb{P}^3$.

<u>e.g.</u> $P(z) = 3z+1$, <u>$\mathrm{Hilb}(P, \mathbb{P}^3)$</u> $= ?$

$\Rightarrow$ $= \{ X \subseteq \mathbb{P}^3,\ P_X = P(z) \}$.

- (1) $\deg(p(z)) = 1$, $\Rightarrow$ X is curve, $\dim X = 1$.

  (2) $\mathrm{lc}(p(z)) = 3$, $\Rightarrow$ $\deg(X) = 3$.

  (3) If X is an integral curve, $g(X) = 0$.

  $\Rightarrow$ X is a rational curve.

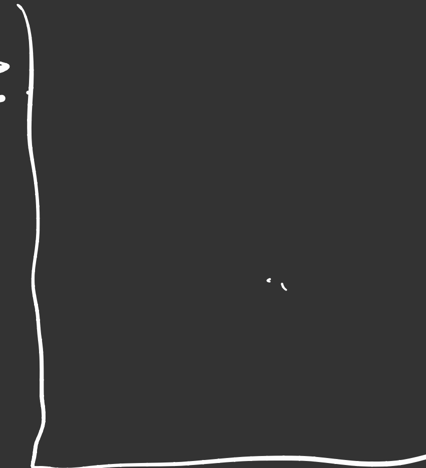
- FACT (I), $\exists$ a component $\mathcal{H}_1 \subseteq \mathrm{Hilb}(p, \mathbb{P}^3)$.

$$\mathcal{H}_1 = \{ \deg(X) = 3,\ X \text{ is rational} \}$$

twisted cubic.

$$\mathbb{P}^1 \to \mathbb{P}^3$$

· **Warning**: this is only one component



(4). If X is not integral,

$$X = X_1 \cup X_2.$$

$$3z+1 = P_X = P_{X_1} + P_{X_2}.$$

$\Rightarrow$ $P_{X_1} = 3z$, $\Rightarrow$ X_1 is a plane cubic, $g(X_1) = 1$.

$P_{X_2} = 1$. X_2 is a point.

$$\mathcal{H}_2 = \{\text{plane cubic} + \text{a point}\} \subseteq \mathrm{Hilb}(P, \mathbb{P}^3)$$

· One last remark:

Castelnuovo Bound for curves in $\mathbb{P}^3$

$$g(C) \leq \begin{cases} \frac{1}{4}\deg^2(C) - \deg(C) + 1, & \deg(C) \text{ even} \\ \frac{1}{4}(\deg(C)^2 - 1) - \deg(C) + 1. & \deg(C) \text{ odd} \end{cases}$$