Algebraic surfaces

Lecture III: minimal models

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Geometrically ruled surfaces

Definition

- A surface S is **ruled** if it is birational to $C \times \mathbb{P}^1$.
- If $C = \mathbb{P}^1$, we say that S is **rational**.
- S is **geometrically ruled** if $\exists\, p : S \to C$ smooth, fibers $\cong \mathbb{P}^1$.

The last definition is justified by:

Theorem (Noether-Enriques)

$p : S \to C$ geometrically ruled $\Rightarrow$ S ruled.

Note that this is specific to surfaces: there exist smooth morphisms $X \to S$ (S surface) with all fibers $\cong \mathbb{P}^1$, but X not birational to $S \times \mathbb{P}^1$ (*Severi-Brauer varieties*).

Minimal ruled surfaces

Theorem

S ruled not rational. S minimal $\Leftrightarrow$ S geometrically ruled.

Proof : 1) $p : S \to C$ with fibers $\cong \mathbb{P}^1$, $g(C) \geqslant 1$.

If $E \subset S$, $p(E) = q \in \mathbb{P}^1$ since $g(C) \geqslant 1 \Rightarrow E = p^{-1}(q) \Rightarrow E^2 = 0$.

2) $S \cong C \times \mathbb{P}^1 \rightsquigarrow$ rational map $p : S \dashrightarrow C$, $g(C) \geqslant 1$.

Claim : p is a morphism.

If not,
$$\begin{tikzcd} & S_n \arrow[dl, "u"'] \arrow[dr, "v"] & \\ S \arrow[rr, dashed, "\varphi"] & & C \end{tikzcd}$$
$u : S_n \to S_{n-1} \to \cdots \to S_0 = S$.

$E_n \subset \hat{S}$ exceptional curve; since $g(C) \geqslant 1$, $v(E_n) = \{\mathrm{pt}\} \Rightarrow$ can replace S_n by S_{n-1}, then ... till $S_0 \Rightarrow$ ■ .

End of the proof

3) $p : S \to C$, general fiber $F \cong \mathbb{P}^1$. Want to prove **all** fibers $\cong \mathbb{P}^1$.
Recall: $F^2 = 0$, $K \cdot F = -2$ (genus formula).

- F irreducible $\Rightarrow$ $F \cong \mathbb{P}^1$ (genus formula).
- $F = mF'$? Only possibility $m = 2$, $K \cdot F' = -1$, contradicts genus formula.
- $F = \sum n_i C_i$. **Claim :** $\Rightarrow$ $C_i^2 < 0 \ \forall i$.

Because: $n_i C_i^2 = C_i \cdot (F - \sum_{j \neq i} n_j C_j)$, $C_i \cdot F = 0$, $C_i \cdot C_j \geqslant 0$, and $C_i \cdot C_j > 0$ for some j since F is connected.

- Then $K \cdot C_i = 2g(C_i) - 2 - C_i^2 \geqslant -1$, $= -1 \Leftrightarrow C_i$ exceptional.

So if S minimal, $(K \cdot C_i) \geqslant 0 \ \forall i \ \Rightarrow \ (K \cdot F) \geqslant 0$, contradiction. ■

Projective bundles

E rank 2 vector bundle on C $\rightsquigarrow$ projective bundle $p : \mathbb{P}_C(E) \to C$, $p^{-1}(x) = \mathbb{P}(E_x)$, so $\mathbb{P}_C(E)$ is a geometrically ruled surface.

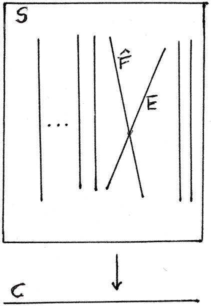
The following can be deduced from the Noether-Enriques theorem:

Proposition

Every geometrically ruled surface is a projective bundle.

There is a highly developed theory of vector bundles on curves, particularly in rank 2; therefore the classification of minimal ruled surfaces is well understood.

Elementary transformation



$f : S \to C$ geometrically ruled. Choose $p \in C$, $q \in F := f^{-1}(p)$. Blow up q.

$\hat{f} : \hat{S} \xrightarrow{b} S \xrightarrow{f} C$. Fiber above $p = E \cup \hat{F}$.

$0 = (\hat{f}^* p)^2 = (E + \hat{F})^2 = E^2 + \hat{F}^2 + 2 \ \Rightarrow$ $\hat{F}^2 = -1$, hence $\hat{F}$ is an exceptional curve (Castelnuovo). Contraction $c : \hat{S} \to S'$:

$$\Longrightarrow \quad \begin{tikzcd} & \hat{S} \arrow[dl, "b"'] \arrow[dd, "\hat{f}"] \arrow[dr, "c"] & \\ S \arrow[dr, "f"'] & & S' \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

$\hat{f}$ induces $g : S' \to C$ geometrically ruled.

Elementary transformation with section

$$\begin{tikzcd} & \hat{S} \arrow[dl, "b"'] \arrow[dr, "c"] & \\ S & & S' \end{tikzcd}$$

Let $\Sigma \subset S$ be a section of f passing through q. Then Σ and F are transverse, so $\hat{\Sigma} \cap \hat{F} = \varnothing$ in $\hat{S}$, and c maps $\hat{\Sigma}$ isomorphically to Σ' section of g.

Then $$\Sigma'^2 = \hat{\Sigma}^2 = (b^*\Sigma - E)^2 = \Sigma^2 - 1\,.$$

Lemma

Suppose $\mathrm{Pic}(S) = \mathbb{Z}[F] \oplus \mathbb{Z}[\Sigma]$. *Then* $\mathrm{Pic}(S') = \mathbb{Z}[F'] \oplus \mathbb{Z}[\Sigma']$.

Proof : It suffices to prove that $(c^*F', c^*\Sigma', E)$ basis of $\mathrm{Pic}(\hat{S})$. But $c^*F' = b^*F$, $c^*\Sigma' = \hat{\Sigma} = b^*\Sigma - E$, and $(b^*F, b^*\Sigma, E)$ basis of $\mathrm{Pic}(\hat{S})$. ∎

The surfaces $\mathbb{F}_n$

Proposition

- *For $n \geqslant 0$, $\exists$ a geometrically ruled rational surface $\mathbb{F}_n \to \mathbb{P}^1$, with a section Σ of square $-n$, and $\mathrm{Pic}(\mathbb{F}_n) = \mathbb{Z}[F] \oplus \mathbb{Z}[\Sigma]$.*
- *For $n > 0$, the curve Σ is the only curve of square < 0 on $\mathbb{F}_n$.*

Proof : We start with $\mathbb{F}_0 := \mathbb{P}^1 \times \mathbb{P}^1$, with $f = \mathrm{pr}_1$ and $\Sigma = \mathbb{P}^1 \times \{0\}$. Once $(\mathbb{F}_n, \Sigma)$ is constructed, we choose $q \in \Sigma$: elementary transformation $\rightsquigarrow$ $\mathbb{F}_{n+1} = S_1$ with $\Sigma^2 = -n - 1$.

- By the Lemma, $\mathrm{Pic}(\mathbb{F}_n) = \mathbb{Z}[F] \oplus \mathbb{Z}[\Sigma]$.
- Let $C \neq \Sigma$ irreducible curve on $\mathbb{F}_n$. $C \equiv a\Sigma + bF$.

$(C \cdot F) \geqslant 0 \ \Rightarrow \ a \geqslant 0$; $\qquad (C \cdot \Sigma) = -an + b \geqslant 0$

$\Rightarrow \ C^2 = -na^2 + 2ab = a(2b - an) \geqslant an^2 \geqslant 0$. ∎

Minimal rational surfaces

Corollary

$\mathbb{F}_n$ is minimal for $n \neq 1$.

$\mathbb{F}_1$ is obtained by blowing up a point q in $\mathbb{P}^1 \times \mathbb{P}^1$ and contracting one of the lines through q; by stereographic projection, $\mathbb{F}_1 \cong \hat{\mathbb{P}}^2$.

Theorem

The minimal rational surfaces are $\mathbb{P}^2$ and $\mathbb{F}_n$ for $n \neq 2$.

Remark : Being geometrically ruled, the surfaces $\mathbb{F}_n$ are of the form $\mathbb{P}_{\mathbb{P}^1}(E)$. It is not difficult to show that all vector bundles on $\mathbb{P}^1$ are direct sums of line bundles; in fact, it was observed by Hirzebruch that $\mathbb{F}_n = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$.