Week 1: Exam Solutions

1. (20 points) Let k be an algebraically closed field of characteristic 0. Assume

$$A = k[x, y, z]/(x^2y + xy^2 + z^3 + 1).$$

And define the affine scheme $S = \operatorname{Spec} A$.

(1) Describe all points of S, as well as the closure of each point. Determine which of them are closed points.

(2) Determine whether S is separated. And determine whether S is proper.

(3) Let X be the subscheme of $\mathbb{P}^3_k$ defined by a single homogeneous polynomial

$$g(x_0, x_1, x_2, x_3) = x_0^3 + x_1^2x_2 + x_1x_2^2 + x_3^3.$$

Show that S is an affine open subscheme of X.

(4) Let C be the intersection of X and the plane $(x_0 = 0)$ in $\mathbb{P}^3_k$, and $\mathcal{O}_C$ its structure sheaf. Compute $\check{H}^i(C, \mathcal{O}_C)$ for all non-negative integers i.

Solution:

(1)

For simplicity, we write $f(x, y, z) = x^2y + xy^2 + z^3 + 1$.

The points of S are in one-to-one correspondence with prime ideals of A, which are moreover in one-to-one correspondence with prime ideals of $k[x, y, z]$ containing f.

The only 2-dimension ideal (minimal ideal) of A is (0). To show (0) is a prime ideal, one has to prove f is an irreducible polynomial. We regard f as a quadratic polynomial in x. If f is reducible, then it can be factored into either

$$f = (xy + g_1(y, z))(x + g_2(y, z)) \quad \text{or} \quad f = h_1(x, y, z)h_2(y, z)$$

where the highest degree of x in $h_1(x, y, z)$ is 2. Both cases can be manually ruled out. Indeed, in the first case, since $g_1 \cdot g_2 = z^3 + 1$, both g_1 and g_2 have to be independent of y (otherwise their product contains y since $k[z]$ is a domain). This then would make $g_1 + y \cdot g_2 = y^2$ impossible; in the second case, h_2 would be a common factor of y, y^2 and $z^3 + 1$, which has to be 1. Therefore f is irreducible hence (0) is a prime ideal. It corresponds to the generic point of S. As every prime ideal contains (0), the closure of this point is the collection of all points.

The 0 dimensional ideals (maximal ideals) of A correspond to maximal ideals of $k[x, y, z]$ containing f. Since k is algebraically closed, the Nullstellensatz confirms that every such maximal ideal is of the form $(x - a, y - b, z - c)$ for some $a, b, c \in k$

Date: July 25, 2020.

such that $f(a, b, c) = 0$. The points corresponding to these ideals are closed points of S. The closure of each of these contains only the point itself.

The remaining points are given by all 1 dimensional ideals of A, which are the generic points of irreducible curves on S. (For this question, it might suffice to say just that much, as there doesn't seem to be a reasonable way to classify these ideals – as least the TA doesn't know how to do it.) The closure of every such ideal contains not only the point itself, but also all the closed points of the curve.

(2)

The scheme S is separated. Indeed, one can prove that every affine scheme $\operatorname{Spec} A$ is separated either by the definition of separatedness in Hartshorne, or by the valuative criterion mentioned in lectures.

First approach: By the definition in Hartshorne, one has to show that the diagonal map $\operatorname{Spec} A \to \operatorname{Spec} A \times \operatorname{Spec} A$ is a closed embedding. Equivalently, the ring homomorphism $A \otimes_k A \to A, (x, y) \mapsto xy$ is surjective, which is obvious.

Second approach: By the valuative criterion, one has to show there is at most one way to fit the dashed arrow to make the diagram

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & \operatorname{Spec} A \\ \big\downarrow & \nearrow & \\ \operatorname{Spec} R & & \end{array}$$

commute, where R is a DVR and K its field of fractions. Dually, this corresponds to the uniqueness of the dashed arrow (if it exists) in the diagram

$$\begin{array}{ccc} K & \longleftarrow & A \\ \big\uparrow & \swarrow & \\ R & & \end{array}$$

which is obvious as $R \to K$ is injective.

The scheme S is not proper. This is intuitively clear. For a rigorous argument, one can invoke either the definition of properness in Hartshorne, or the valuative criterion mentioned in lectures.

First approach: Using the definition in Hartshorne, we can check that the structure morphism $\operatorname{Spec} A \to \operatorname{Spec} k$ is not universally closed. There are many ways to see this. For instance, it is easy to see that the affine line $L = \operatorname{Spec} k[x, y, z]/(x, z + 1)$ is a closed subscheme of S. We claim the projection morphism $S \times L \to L$ is not closed. In fact, let H be the closure of the set $\{((0, y_1, -1), (0, y_2, -1)) \mid y_1 y_2 = 1\}$ in $S \times L$ (which is actually defined by 3 equations in $S \times L$). The image of H under the second projection to L contains all points except the origin, which is not closed.

Second approach: With use the valuative criterion, we consider the diagram

$$\begin{array}{ccc} \operatorname{Spec} k(t) & \longrightarrow & \operatorname{Spec} A \\ \big\downarrow & \nearrow & \\ \operatorname{Spec} k[t]_{(0)} & & \end{array}$$

which dually corresponds to

$$\begin{array}{ccc} k(t) & \longleftarrow & k[x,y,z]/(f) \\ \uparrow & \swarrow & \\ k[t]_{(0)} & & \end{array}$$

in which $x \mapsto 0$, $y \mapsto \frac{1}{t}$ and $z \mapsto -1$ in the horizontal arrow. It is clear that the dashed arrow cannot exist to make the diagram commute, as the vertical arrow is injective and $\frac{1}{t}$ is not in its image.

(c)

$\mathbb{P}^3_k$ can be covered by standard affine charts U_i where $0 \leqslant i \leqslant 3$. The part of X in $U_0 = (x_0 \neq 0)$ is precisely S.

(d)

This is essentially Exercise IV-4.

The curve C lives in $\mathbb{P}^2_k$ (defined by $x_0 = 0$) and avoids the point $[0:0:1]$. Therefore we can cover it by two affine charts U_1 and U_2, where

$$\begin{aligned} U_1 &= C \cap (x_1 \neq 0) = \operatorname{Spec} k[u,t]/(f(u,1,t)); \\ U_2 &= C \cap (x_2 \neq 0) = \operatorname{Spec} k[ut^{-1},t^{-1}]/(f(ut^{-1},t^{-1},1)). \end{aligned}$$

The Čech complex for computing $\check{H}(C, \mathcal{O}_C)$, namely

$$\Gamma(U_1, \mathcal{O}_{U_1}) \oplus \Gamma(U_2, \mathcal{O}_{U_2}) \longrightarrow \Gamma(U_1 \cap U_2, \mathcal{O}_{U_1 \cap U_2})$$

can be explicitly written as

$$\begin{array}{ccc} k[u,t]/(f(u,1,t)) \oplus k[ut^{-1},t^{-1}]/(f(ut^{-1},t^{-1},1)) & \xrightarrow{\xi} & k[u,t,t^{-1}]/(f(u,1,t)) \\ (g(u,t) \quad , \quad h(ut^{-1},t^{-1})) & \longmapsto & g(u,t) - h(ut^{-1},t^{-1}) \end{array}$$

where $f(u,1,t) = u^3 + t^2 + t$.

First we analyze $\ker \xi$. Assume $g(u,t) - h(ut^{-1},t^{-1}) = f(u,1,t) \cdot \varphi(u,t,t^{-1})$. Then φ can be regarded as a Laurent polynomial in t with coefficients in $k[u]$. Without loss of generality, we can assume φ does not contain terms with non-negative powers of t, as otherwise we can choose a different representative of g to make this happen. Hence every term in φ is a fraction. Moreover, we can also assume every term in φ is of degree strictly larger than -3, as otherwise we can replace h by a different representative to make this happen. Hence every term in φ has a certain power of t in its denominator and is of degree larger than -3. If such a function $\varphi \neq 0$, then we look at its terms with the highest power of u, say $u^k \varphi_0(t^{-1})$. Then the terms with the highest power of u in the product $f \cdot \varphi$ is $u^{k+1}\varphi_0(t^{-1})$, in which each term is of positive degree. This is a contradiction as neither $g(u,t)$ nor $h(ut^{-1},t^{-1})$ contains such a term. Therefore $\varphi = 0$, which implies $g(u,t) = h(ut^{-1},t^{-1})$, which is possible only when g and h are the same constant function. This proves that $\check{H}^0(C, \mathcal{O}_C) = \ker \xi \cong k$.

Then we compute $\operatorname{coker} \xi$. Due to the form of $f(u,1,t)$, every element in the target $k[u,t,t^{-1}]/(f(u,1,t))$ can be represented by a Laurent polynomial, in which no term is divisible by u^3. It is clear that every term without a denominator can be hit by the image of $k[u,t]/(f(u,1,t))$, and every term of non-positive degree can be hit by the image of $k[ut^{-1},t^{-1}]/(f(ut^{-1},t^{-1},1))$. The only remaining term which could appear in the above Laurent polynomial is a constant multiple of $\frac{u^2}{t}\varphi_1(u)$. This proves that $\check{H}^1(C,\mathcal{O}_C) = \operatorname{coker}\xi \cong k$.

Since the Čech complex contains only two non-trivial terms, we immediately obtain $\check{H}^i(C,\mathcal{O}_C) = 0$ for all $i \neq 0$ or 1.

2. (20 points) Let G, B, T be a reductive group, a Borel subgroup, and a maximal torus, and let W be the Weyl group. The action of W on $X(T)$ extends to an action on the group ring $Z[X(T)]$, and so one can consider the ring of invariants $Z[X(T)]^W$. Show that for any finite-dimensional G-representation V , the formal character $ch(V)$ lies in $Z[X(T)]^W$.

Proof. Assume that (V, π) is a finite dimensional representation of G. Then we have a decomposition according to the weight : $V = \bigoplus_{\lambda \in P(\pi)} V_\lambda$, where $P(\pi) = \{\lambda \in X(T) | V_\lambda \neq 0\}$. Now for any $w \in W$, take $\tilde{w} \in G$ which represents w. Consider the representation: $(V, \pi^{\tilde{w}})$ by :

$$\pi^{\tilde{w}}(g) = \pi(\tilde{w}^{-1} g \tilde{w})$$

for any $g \in G$. For $t \in T$ and $v \in V_\lambda(\pi)$, we have:

$$\begin{aligned} \pi^{\tilde{w}}(t)v &= \pi(\tilde{w}^{-1} t \tilde{w})v \\ &= \lambda(\tilde{w}^{-1} t \tilde{w})v \\ &= w(\lambda)(t)v \end{aligned}$$

This implies that $v \in V_{w(\lambda)}(\pi^{\tilde{w}})$, hence

$$V = \bigoplus_{\lambda \in P(\pi)} V_{w(\lambda)}$$

But we have an isomorphism of represnetations :

$$\begin{aligned} \varphi : (V, \pi) &\to (V, \pi^{\tilde{w}}) \\ v &\mapsto \pi(\tilde{w}^{-1})v \end{aligned}$$

Therefore the two representations share the same weight space,i.e,

$$w(P(\pi)) = P(\pi)$$

for any $w \in W$. This shows that:

$$ch(V) = \sum_{\lambda \in P(\pi)} dim(V_\lambda) e^\lambda$$

is invariant under W. □

3. (20 points) Let $k = \bar{k}$ be an algebraic closed field. Hurwitz's theorem states that if char(k)= 0 and C is a non-singular projective curve over k of genus $g(C) \geq 2$, then

$$|\mathrm{Aut(C)}| \leq 84(\mathrm{g} - 1)$$

1. Assume that char(k)= 0 and $g(C) = 2$, show that C cannot have an automorphism of order 7. Conclude that Hurwitz's bound is not optimal when $g(C) = 2$.

2. Assume that char(k)= $p > 0$. Let C be the non-singular projective model of the plane curve $\{Y^2 = X^p - X\} \subset \mathbb{A}^2_k$. Compute the genus of C and prove that for p sufficiently large, $|\mathrm{Aut(C)}| > 84(\mathrm{g} - 1)$. The implies Hurwitz's theorem fails in positive characteristics.

Proof. 1. Assume that there is a group $G \subset \mathrm{Aut}(X)$ and $|G| = 7$. We consider the degree of ramification divisor R to derive contradiction. Let C' be the normalization of C/G. Since C is normal, the quotient C/G is also normal hence nonsingular and the quotient map $\pi : C \to C'$ has degree 7. If $c \in C$ is a closed point, we have $|G \cdot c| \cdot |G_c| = |G| = 7$, so $|G \cdot c| = 1$ or 7. That means for a closed point $c' \in C'$, the inverse image $\pi^{-1}(c')$ consists of 1 or 7 points, hence the ramification index $e_c = 0$ or 7. Therefore, degree of the ramification divisor R must be divisible by 6. By Hurwitz's theorem, we have $2g(C) - 2 = deg(\pi)(2g(C') - 2) + \deg R$. If $g(C') \geq 2$, we have $\deg R < 0$, which is impossible. If $g(C') = 0$ or 1, then $\deg R = 2$ or 16, which are not divisible by 6.

2. If $\mathrm{char}(k) = 2$, the curve is a smooth conic, genus 0. Now we consider the case $\mathrm{char}(k) \neq 2$. Let C be the normalization of the curve. Let $\pi \colon C \to \mathbb{P}^1, [X : Y : Z] \mapsto [X : Z]$ be the projection. Over $\mathbb{A}^1$, it is ramified at p points with the ramification index 2 . It is also possibly ramified at ∞, with ramification index $e_p \leq 2$. By Hurwitz formula, $2g(C) - 2 = 2(0-2) + \sum(e_P - 1) + (e_\infty - 1)$. Therefore $2g(C) + 2 = p + e_\infty - 1$, the genus is an integer, and $e_\infty \in \{1, 2\}$, it forces $e_\infty = 2$ and $g(C) = \frac{p-1}{2}$.

   Let $a_1, \ldots, a_p$ be the roots of $x^p - x$ and $j = 0, 1, \ldots, p-1$. We consider the following two kinds of automorphism of $\mathbb{A}^2_k$:

$$\begin{cases} x \mapsto a_i^2 x \\ y \mapsto a_i y \end{cases} \quad \begin{cases} x \mapsto x + j \\ y \mapsto y \end{cases} \tag{1}$$

   They both fix the curve C and generate an order $p(p-1)$ subgroup in $\mathrm{Aut}(C)$. So $|\mathrm{Aut}(C)| \geq p(p-1)$. Since $g(C) = \frac{p-1}{2}$, we get the conclusion.

□

4. (20 points) Let R be a ring and $x_1, \cdots, x_n$ be a regular sequence, denote $I = (x_1, \cdots, x_n)$.

1. Given $x \in R$, show that if $(I : x) = I$, then $(I^d : x) = I^d$ for any $d > 0$.
2. Show that if $f(\underline{t}) \in R[t_1, \cdots, t_n]$ homogeneous degree d and $f(\underline{x}) \in I^{d+1}$, then $f \in I[\underline{t}]$.

Proof. We split the proof into two steps

- $(2_{d,n}) \Rightarrow (1_{d,n})$ For $d = 1$, there's nothing to prove, let's assume $d \geq 2$. For any $a \in (I^d : x)$, we have $ax \in I^d \subset I$. By assumption $(I : x) = I$, we know $a \in I$. Therefore we can write $a = \sum r_i x_i$. Note that $\sum (r_i x) x_i = ax \in I^d \subset I^2$. By (2) we know $r_i x \in I$, again by $(I : x) = I$, we know $r_i \in I$. Therefore we can write $r_i = r_{ij} x_j$ and $a = \sum r_{ij} x_i x_j$. We repeat the process $(d-2)$ times and conclude.

- $(2_{d-1,n}) + (2_{d,n-1}) \Rightarrow (2_{d-1,n}) + (1_{d,n-1}) \Rightarrow (2_{d,n})$. Let $J = (x_1, \cdots, x_{n-1})$, we know $(J : x_n) = J$. Let's write $f = g + t_n h$, where $g \in (R[t_1, \cdots, t_{n-1}])_d$ and $h \in (R[t_1, \cdots, t_n])_{d-1}$. A clever observation is that we can assume $f(\underline{x}) = 0$. (Given $f(\underline{x}) \in I^{d+1}$, there exists a homogeneous polynomial $k \in (R[t_1, \cdots, t_n])_{d+1}$ such that $k(\underline{x}) = f(\underline{x})$. We write $k = \sum t_i k_i$ where $k_i \in (R[t_1, \cdots, t_n])_d$. Then we may replace f by $f(\underline{t}) - \sum x_i k_i(\underline{t})$). Then $g(\underline{x}) + x_n h(\underline{x}) = 0$. By $(1_{d,n-1})$ we know $h(\underline{x}) \in (J^d : x_n) = J^d \subset I^d$. Applying $(2_{d-1,n})$ we know $h \in I[\underline{t}]$. Since $h(\underline{x}) \in J^d$, there exists $l(x) \in R[t_1, \cdots, t_{n-1}]$ such that $h(\underline{x}) = l(\underline{x})$. Putting $G(\underline{t}) = g(t_1, \cdots, t_{n-1}) + x_n l(\underline{t})$ we have $G(\underline{x}) = 0$, by $(2_{d,n-1})$ we know $G \in I[\underline{t}]$, therefore $g \in I[\underline{t}]$. Then $f = g + x_n h \in I[\underline{t}]$.

- We prove $(2_{d,n})$ by induction on $(d, n) \in \mathbb{N} \times \mathbb{N}$, the initial cases $(0, n)$ and $(d, 1)$ are clear.

$\square$