Introduction to algebraic curves

1) Generalities of algebraic varieties

Hartshorne Chapter I 1-6

2) Nonsingular curves

3) divisors, canonical divisors, genus of curves

4) Riemann-Roch, automorphisms of curves

5) special divisors, embedding of curves, parametrizing curves

$k = \bar{k}$.

affine spaces: $\mathbb{A}_k^n = \{(a_1, \dots, a_n) \mid a_i \in k\}$

Zariski topology: $A = k[x_1, \dots, x_n]$ polynomial ring.

closed subsets:

$$V(I) = \{P \in \mathbb{A}_k^n \mid f(P) = 0 \ \ \forall f \in I\}$$

$I \subset A$ ideal.

affine variety: irreducible closed subsets of $\mathbb{A}^n_k$.

Hilbert's Nullstellensatz: $Y \subset \mathbb{A}^n_k$ affine variety

$$\Updownarrow$$

$I \subset A = k[x_1, \dots, x_n]$ prime ideal

$$Y \longmapsto I(Y) = \{ f \in A \mid f(P) \ \forall P \in Y \}$$

$$I \longmapsto Z(I) = \{ P \in \mathbb{A}^n_k \mid f(P) = 0 \ \forall f \in I \}$$

Projective space: $\mathbb{P}^n_k = \{(a_0 : a_1 : \cdots : a_n) \mid a_i \text{ not all zero}\}$

$(a_0 : a_1 \cdots : a_n) = (b_0 : b_1 : \cdots : b_n)$

$\Leftrightarrow \exists \lambda \in k^*$

$a_i = \lambda b_i$.

$\rightsquigarrow S = k[x_0, \cdots, x_n]$

$\hookrightarrow$ graded ring.

Zariski topology: Closed subsets:

$I \subset S$ homogeneous ideal

$V(I) = \{P \in \mathbb{P}^n_k \mid f(P) = 0,\ \forall f \in I\}$

irreducible closed subsets of $\mathbb{P}^n_k$ are called

projective varieties.

projective varieties in $\mathbb{P}^n_k$

$\Updownarrow$

prime homogeneous ideals of S

Rmk projective variety $\rightsquigarrow$ compact manifold

$\mathbb{P}^n_k \rightsquigarrow n+1$ affine charts:

$$U_i = \{ (a_0, a_1, \cdots a_{i-1}, 1 \quad a_{i+1} \ldots a_n) \} \text{ open subset of } \mathbb{P}^n_k.$$

$$\simeq \mathbb{A}^n_k.$$

Projective varieties are covered by affine varieties

Open subsets of projective variety are called quasi-projective variety.

(affine) (affine).

Example $\mathbb{P}^2_k \setminus P = (1:0:0)$ quasi-proj variety (not affine)

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Morphism

Def X affine variety $\subseteq \mathbb{A}^n_k$. $f: X \mapsto k$. function.

f regular at $P \in X$ if $\exists$ $U \ni P$ an open subset. such that $f|_U = \frac{g}{h}$.

$g, h \in A = k[x_1, \dots, x_n]$.

$h(Q) \neq 0, \ \forall Q \in U$.

f is regular if it is regular at each point of X.

$\mathcal{O}(X)$ the set of regular functions.

$\hookrightarrow$ ring.

$\forall P \in X$

$\mathcal{O}_{X,P}$ germs of regular functions at P

$$= \left\{ (U, f) \;\middle|\; \begin{array}{l} U \ni P \text{ open subset.} \\ f \in \mathcal{O}(U). \end{array} \right\} \Big/ \sim$$

$\hookrightarrow$ local ring.

maximal ideals: $\{(U,f) \mid f(P)=0\}$

$(U_1, f_1) \sim (U_2, f_2)$ if $f_1 = f_2|_{U_1 \cap U_2}$.

$$K(X) = \{(U,f) \mid U \subset X \text{ open}, \ f|_U \text{ regular}\}$$

Prop $X \hookrightarrow \mathbb{A}^n_k$ affine. $\rightsquigarrow I(X)$

- $\mathcal{O}(X) = A/I(X) =: A_X \rightsquigarrow$ domains
- $\forall P \in X$, $\mathcal{O}_{X,P} = \mathcal{O}(X)_{m_P}$. m_P the maximal ideal corresponding to P.
- $K(X) = Q(A_X)$ the quotient field of A_X.

Given X, Y two varieties.

we say $f: X \mapsto Y$ is a $\boxed{\text{morphism}}$ between X and Y

if. (1) f continuous map.

(2) $\forall$ V open subset of Y.

$g: V \mapsto k$ regular.

then

$$\begin{tikzcd} f^{-1}(V) \arrow[r, "f"] \arrow[rr, bend right] & V \arrow[r, "g"] & k \end{tikzcd}$$

should be regular

Prop X variety, Y affine variety.

$$\mathrm{Hom}(X, Y) \longleftrightarrow \mathrm{Hom}(A_Y, \mathcal{O}(X)).$$

$\mathrm{Hom}(X,Y)$ $\downarrow$ the set of morphisms between X & Y.

$A_Y = \mathcal{O}(Y)$

Rational map X, Y varieties.

a rational map $\varphi : X \dashrightarrow Y$. is an equivalence of classes

(U, φ_U).

$U \subset X$ non-empty open subset.

$\varphi_U : U \longrightarrow Y$. morphism.

$(U, \varphi_U) \sim (V, \varphi_V) \Leftrightarrow \varphi_U = \varphi_V$ on $U \cap V$.

a rational map (U, φ_U) is $\boxed{\text{dominant}}$ if $\varphi_U(U)$ is dense in Y.

X, Y varieties

<u>Prop</u> $\{\varphi: X \dashrightarrow Y \text{ dominant rational map}\}$

$$\Updownarrow$$

$\{k\text{-algebra homomorphism } K(Y) \hookrightarrow K(X)\}$

Rmk Any $\underbrace{\text{variety } X \text{ of dimension } n}$

is birational to a hypersurface T in $\mathbb{P}_k^{n+1}$

$\{g = 0\}$, $g \in S$ homogeneous.

$\varphi_U: X \dashrightarrow Y \hookrightarrow \mathbb{P}_k^{n+1}$

$\Leftrightarrow K(X) \simeq K(T)$, $k = \bar{k}$.

$\exists\ \varphi_V: T \dashrightarrow X$ inverse map

$\varphi_V \circ \varphi_U = \mathrm{Id}_X \quad \forall\ \varphi_U$

$\varphi_U \circ \varphi_V = \mathrm{Id}_T$

Proof: $K(X)/k$ separated generated.

$\exists\ x_1, \dots, x_n$ transcendental basis s.t. $K(X)/k(x_1, \dots, x_n)$ separated extension

$\rightsquigarrow \quad K(X) = k(x_1, \ldots, x_n)[y]$

minimal polynomial of y is g.

$\rightsquigarrow \quad K(X) = Q(B)$ $\qquad B = k[x_1, \ldots, x_n, x_{n+1}] / (g)$

dimension: X variety.

$$\mathrm{Tr}(K(X)/k) \rightsquigarrow \dim X = \mathrm{Tr}(K(X)/k)$$

Ex $\mathbb{A}^n_k$ $K(X) = k(x_1, \dots x_n)$ $\dim \mathbb{A}^n_k = n$,

$\boxed{\text{Krull dimension}}$ $\sup\{n \mid Z_0 \subsetneq Z_1 \subsetneq \dots \subsetneq Z_n,$ Z_i irred closed subsets of $X\}$

- Let $X \subseteq \mathbb{A}^n_k$ be an affine variety of dim m.

$I(X) = (f_1, \dots, f_r)$ $\quad f_i \in k[x_1, \dots, x_n]$.

We say X is non-singular at $P \in X$

if $\left(\dfrac{\partial f_i}{\partial x_j}(P) \right)_{\substack{1 \le i \le n \\ 1 \le j \le n}}$ has rank. $n-m$.

We say X is non-singular, if it is non-singular at every point

Theorem. X variety of dimension n. $P \in X$.

X nonsingular at P.

$\Longleftrightarrow$ $\mathcal{O}_{X,P}$ is a regular local ring

i.e. $\dim_k m_P / m_P^2 = n$.

$\cup$

m_P the maximal ideal.

Thm. X variety. $\mathrm{Sing}(X) := \{ p \in X \mid X$ ~~not. non~~singular at $p \}$

Then $\mathrm{Sing}(X)$ is a proper closed subset of X.

Idea.

$$\begin{array}{ccccc} X & \overset{\text{biratnl}}{\dashrightarrow} & Y & \subset & \mathbb{P}^{n+1} \\ \text{open} \uparrow & & \uparrow \text{open} & & \\ U & \xrightarrow{\sim} & V & & \end{array}$$

defined by $g(x_0, \ldots, x_n) = 0$.

$\mathrm{Sing}(Y) = \langle \frac{\partial g}{\partial x_i} \mid 0 \le i \le n \rangle$.

Curves: varieties of dimension 1

Non-singular projective curves

C:

- $y^2 = x^3 \subseteq \mathbb{A}_k^2$.

C singular at $(0, 0)$

Thm $\forall$ curve C/k.

$\exists!$ $\pi: \tilde{C} \to C$ a birational morphism, $(K(\tilde{C}) = K(C))$

such that. $\tilde{C}$ is a non-singular curve.

Rmk - Normalization.

- In higher dimension, char $k=0$, Hironaka. established the general theory to find a nonsingular models of a variety

Def. A descrete valuation ring. (DVR) is a Noetherian integral domain R. with a single maximal ideal $=(t)$ that is principal.

$K = Q(R)$

descrete valuation: $v: K \setminus 0 \to \mathbb{Z}$.

Def. Let $S \subset S'$ be a ring extension.

$\forall a \in S'$, a is integral over S

if. a satisfies a monic equation:

$$a^n + a_1 a^{n-1} + \cdots + a_n = 0, \quad a_i \in S.$$

Def Let S be an integral domain. the normalization of S is the integral closure of S in $Q(S)$

$Q(S)$ quotient field of S

integral closure in $Q(S)$: $\{ a \in Q(S) \mid a$ integral over $S \}$

If the integral closure of S is itself

then we say S is normal.