Algebraic Curve IV

Canonical divisor

K/k function field of dim 1.

$\Omega_k(K)$ module of differential: $df / \langle\ ;\ \rangle$

Prop

- $\dim_K \Omega_k(K) = 1$
- If $\operatorname{char} k = 0$, $\forall x \in K \setminus k$, $K\,dx = \Omega_k(K)$
- If $(\mathcal{O}, m)$ a DVR of K, t uniformizing parameter of $(\mathcal{O}, m)$

$$K \cdot dt = \Omega_k(K).$$

$\hookrightarrow \quad \forall\, \omega \in \Omega_k(K)$ $\qquad K \rightsquigarrow C$ non-singular projective curve. $/k$.

$$\mathrm{div}(\omega) := \sum_{P \in C} a_P P. \quad \rightsquigarrow \text{a canonical divisor of } C.$$

let t_P be the uniformizing parameter of $(\mathcal{O}_{C,P}, m_P)$

$\omega = f_P \cdot dt_P$ $\qquad f_P \in K.$ $\qquad a_P := v_P(f_P) \in \mathbb{Z}.$

$\hookrightarrow$ can be written down.

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a canonical divisor is well-defined.

Prop. Let $(\mathcal{O}, m)$ a DVR of K. t a uniformizing parameter of $(\mathcal{O}, m)$. If $f \in \mathcal{O}$.

(Fulton algebraic curve. last chapter.)

then $\dfrac{df}{dt} \in \mathcal{O}$.

$\left(df \in \Omega_k(K) \right.$
$\langle dt \rangle = \Omega_k(K)$

$\left(\dfrac{df}{dt}\right) \cdot dt = df$

$\dfrac{df}{dt} \in K$

$\rightsquigarrow \quad f = f_p\, dt_p$

u_p another uniformizing parameter.

$g \in \mathcal{O}^*_{C,P}$

$f = \tilde{f}_p\, du_p, \quad \tilde{f}_p = g \cdot f_p.$

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- Only finitely many a_p are nonzero.

Example:

$$C = \mathbb{P}^1_k = \{ (x_0 : x_1) \mid x_0, x_1 \in k \}$$

$d\frac{1}{t} = -\frac{1}{t^2} dt$

U_1 $\quad dt = -t^2 \cdot d(\frac{1}{t})$

$$K(C) = k(t) \qquad t = \frac{x_1}{x_0}$$

$$\mathrm{div}(dt) = -2 \cdot P_0$$

$\forall\ P \in U_0$ $\quad \mathcal{O}_{C,P}$

$= (1, a)$

$$C = U_0 \cup U_1$$

$(1 : \frac{x_1}{x_0} = t)$ $\quad (\frac{x_0}{x_1} = \frac{1}{t}, 1)$

$dt = d(t-a)$ $\quad k[t]_{m_p}$ $\quad m_p = (t-a)$

$(0:1) = P_0$

char $k \neq 2, 3$

Example. C $\{Y^2Z = X(X-Z)(X-2Z)\} \subseteq \mathbb{P}^2_k$ $(X:Y:Z)$

C non-singular.

$$\mathrm{div}\left(\frac{X}{Z}\right) = P_0 + P_1 + P_2 - 3Q.$$

U_0 $\{Z \neq 0\}$ $\left(\frac{Y}{Z}\right)^2 = \underbrace{\left(\frac{X}{Z}\right)\left(\frac{X}{Z}-1\right)\left(\frac{X}{Z}-2\right)}_{G}$

$\frac{Y}{Z} = 0$, $\frac{X}{Z} = 0, 1, 2.$ $P_0, P_1, P_2.$ $Q := (0 : 1 : 0)$

$2\frac{Y}{Z}\, d\frac{Y}{Z}$

$= \frac{\partial G}{\partial \frac{X}{Z}} \cdot d\left(\frac{X}{Z}\right).$

Rmk $\omega' \in \Omega_k(K)$ $\exists f \in K.$ $\omega' = f \cdot \omega$.

$$\mathrm{div}(\omega') = \mathrm{div}(f) + \mathrm{div}(\omega).$$

$\hookrightarrow$ $\mathrm{div}(\omega') \sim \mathrm{div}(\omega)$.

Def 1. the genus of C is $g(C)$

$$\ell(\underbrace{\mathrm{div}(\omega)}_{\mathrm{div}(\omega')}) = \dim_k \Gamma(\mathrm{div}(\omega)).$$

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Example! $g(\mathbb{P}^1) = 0 = \ell(-2P_0)$

2. $\deg(\operatorname{div}(\omega)) = 0 \rightsquigarrow g(C)$?

Riemann-Roch Let C be a non-singular projective curve. let K_C be a canonical divisor of C. then for any divisor D on C. we have.

$$\ell(D) - \ell(K_C - D) = 1 + \deg D - g(C)$$

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↳ Let $D = K_C$.

$$\underbrace{\ell(K_C)}_{= g(C)} - \overbrace{1}^{= \ell(0)} = 1 + \deg K_C - g(C)$$

$$\Longrightarrow \boxed{\deg K_C = 2g(C) - 2.}$$

<u>Rmk</u>. In higher dimensions, we have various generalisation of Riemann-Roch,

$k = \mathbb{C}$, Hirzebruch applied topology to generalize RR, to higher dimension non-sing proj. varieties

In general, Grothendieck gave a pure algebraic vast generalization of RR. (Fulton, Intersection theory)

Idea of proof

<u>Method 1</u>. Find a plane curve model D with only node singularity of C

Fulton algebraic curves

<u>Method 2</u> Sheaf and cohomology.

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$$\mathrm{Cl}(C) \overset{\sim}{\longleftrightarrow} \mathrm{Pic}(C) \quad \{\text{Line bundles}\}/\cong$$

$$\mathrm{Div}(C)/\sim \qquad\qquad \text{invertible sheaves}\}$$

$\forall$ effective divisor $D \longmapsto \left(\underset{\substack{\cap \\ C}}{U_i} = \mathrm{Spec}\, R_i \rightsquigarrow \frac{1}{f_i} R_i \subset K(C) \right)$

affine open subset

$\mathcal{O}_C(D) = \left(U_i = \mathrm{Spec}\, R_i \rightsquigarrow \frac{1}{f_i} R_i \subset K(C) \right)$

$\frac{1}{f_i} R_i \to R_i$-module.

$D|_{U_i}$ defined by $f_i = 0$, $f_i \in R_i$.

$D = D_1 - D_2 \rightsquigarrow \mathcal{O}_C(D) = \mathcal{O}_C(D_1) \otimes \mathcal{O}_C(D_2)^{-1}$

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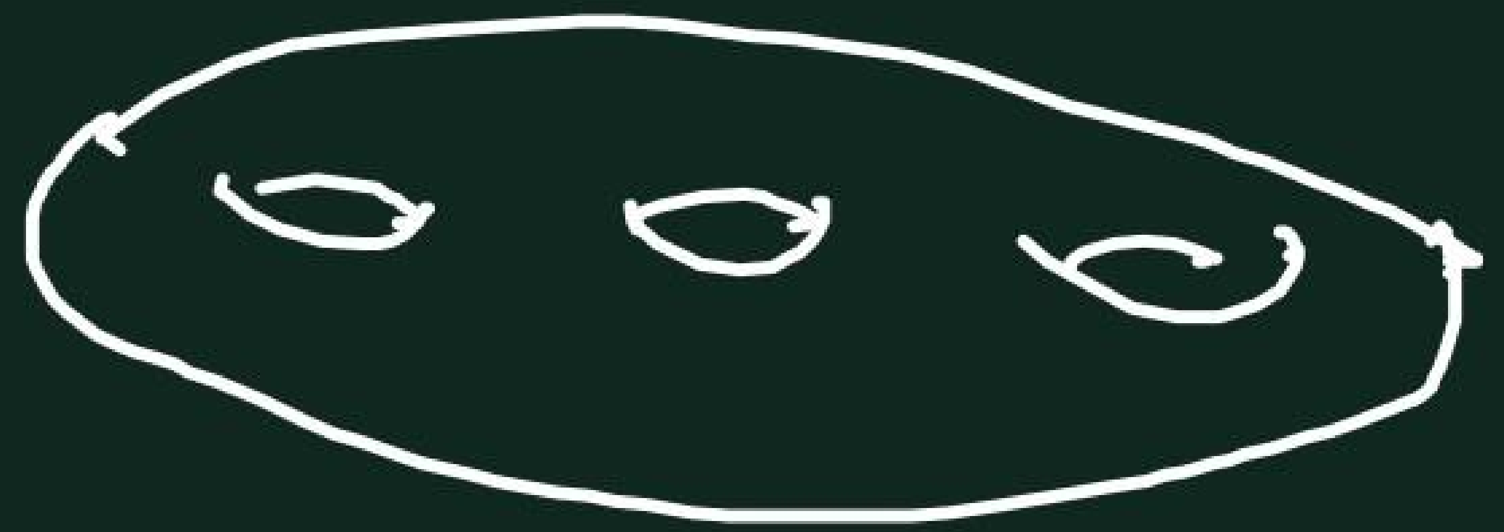
Rmk If $k=\mathbb{C}$, C non-sing proj curve

$\Updownarrow$

compact Riemann surface

$S \rightsquigarrow H^1(S,\mathbb{Z}) \simeq \mathbb{Z}^{2g(S)}$

then $g(C)=g(S)$



$$\Gamma(D) \xrightarrow{\cong} H^0(\mathcal{O}_C(D))$$

$$\ell(K_C - D) \quad ?$$

$$= \dim_k \Gamma(K_C - D) = \dim_k H^0(\mathcal{O}_C(K_C) \otimes \mathcal{O}_C(D)^{-1})$$

Serre duality

$$H^0(\mathcal{O}_C(K_C) \otimes \mathcal{O}_C(D)^{-1}) \cong H^1(\mathcal{O}_C(D))^{\vee}.$$

left side of RR:

$$\ell(D) - \ell(K_C - D) = \dim_k H^0(\mathcal{O}_C(D)) - \dim_k H^1(\mathcal{O}_C(D))$$

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$0 \to \mathcal{O}_C(D) \to \mathcal{O}_C(D+P) \to k_P \to 0$ $\rightsquigarrow$ long exact sequence

$$\hookrightarrow \dim_k H^0(\mathcal{O}_C(D+P)) - \dim_k H^1(\mathcal{O}_C(D+P))$$

$$= \left(\dim_k H^0(\mathcal{O}_C(D)) - \dim_k H^1(\mathcal{O}_C(D))\right) + 1.$$

then by Induction on $\deg D$ $\quad$ ($\dim H^0(\mathcal{O}) - \dim H^1(\mathcal{O}) = 1-g$) $\quad$ $\dim H^0(k_C) = 1$

$\rightsquigarrow$ RR.

$\mathbb{P}^1 \rightsquigarrow g(\mathbb{P}^1)=0$

Let C be a non-sing proj curve of genus 0

$\forall\ P \neq Q \in C$ $D = P - Q$, $\deg D = 0$.

$\hookrightarrow$ RR: $\ell(D) - \overset{0}{\overset{\shortparallel}{\ell(K_C - D)}} = 1 + d - g = 1 \rightsquigarrow \ell(D) = 1$

$\deg K_C = 2g-2 = -2 < 0$ $(f)_0 - (f)_\infty$

$\exists f \in K(C)$ s.t. $\operatorname{div}(f) + D \geq 0$ $\overset{\shortparallel}{\operatorname{div}(f)} + P - Q \geq 0$

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$\Rightarrow \quad \mathrm{div}(f) = Q - P.$

$(f)_0 = Q \qquad (f)_\infty = P.$

$[K(C) : K(f)] = \deg (f)_0 = 1.$

$\Rightarrow \quad K(C) = K(f)$ and $C \xrightarrow{\sim} \mathbb{P}^1$

<u>Example 2.</u> C non-sing proj curve of genus 1.

$\deg K_C = 0. \qquad \ell(K_C) = 1 \qquad K_C \sim 0$

$Cl(C) = \mathrm{Div}(C)/\sim$

$Cl^0(C) = \deg 0$ part of $Cl(C)$

↳ a subgroup of $Cl(C)$.

Fix a point $P_0 \in C$.

then $\forall P \in C$. $D_P := P - P_0$ a divisor of deg 0.

$\rightsquigarrow \phi: C \xrightarrow{\sim} Cl^0(C)$ (C called an elliptic curve)

$P \longmapsto [D_P]$

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Hurwitz's theorems

Given $f: C \to D$ a non-trivial morphism between two non-singular proj curves

$\Updownarrow$

$K(D) \hookrightarrow K(C)$

$\deg f := [K(C) : K(D)]$

We say f is separable if $K(C)/K(D)$ separable

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$\forall P \in C \rightsquigarrow$ ramification index of f at P. e_P

let $Q = f(P)$. $\quad t_Q$ uniformizing parameter of $(\mathcal{O}_{D,Q}, m_Q)$

$$e_P = v_P(t_Q)$$

If $e_P > 1$, we say f ramified at P,

Q a branch point of f.

If char $k=0$. the ramification is tame.

char $k=p$ and $p \nmid e_p$, the ramification is tame

($p \mid e_p$, . - - is wild)

<u>Thm 1</u> (Hurwitz)

Let $f: C \longrightarrow D$ be a separable morphism^{+} (always tame ramification)

between non-singular proj curves. Then,

$$2g(C)-2=(\deg f)\cdot[2g(D)-2]+\sum_{P\in C}(e_P-1),$$

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Proof. (sketch.)

$2g(C)-2 = \deg(\mathrm{div}(w))$ $w \in \Omega_k(K(C))$

- $C \xrightarrow{f} D$ separable. $\Leftrightarrow$ $K(C)/K(D)$ separable

morphism.

take. $f \in K(D)$ $\mathrm{div}(df)$ a canonical divisor of D.

$\deg \mathrm{div}(df) = 2g(D)-2$

$\mathrm{div}(df) = \sum a_Q Q$

QED.

$df = t_Q^{a_Q} \cdot dt_Q$

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Since $K(C)/K(D)$ separable $\rightsquigarrow$ $\overset{\text{see}}{f}$ an element of $K(C)$

df is a generator of $\Omega_k(K(C))$.

Now compute $\operatorname{div}(df)$ on C.

$\forall P \in C$. $\quad t_Q = t_P^{e_P} (*)$

$Q = f(P)$. $\quad df = t_Q^{a_Q} \, dt_Q$

$$\left.\begin{array}{l} t_Q = t_P^{e_P} (*) \\ df = t_Q^{a_Q} \, dt_Q \end{array}\right\} \Longrightarrow df = t_P^{e_P \cdot a_Q} \cdot \left[t_P^{e_P - 1} (*) \, dt_P \right]$$

$v_P(df) = e_P \cdot a_Q + e_P - 1.$

$\forall Q \in D$

last lecture $\deg f = \sum_{P \in f^{-1}(Q)} e_P$

$\deg(f)_\omega = \deg f^*(0)$

$\mathbb{P}^1$

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Thm2 (Hurwitz)

Char $k = 0$

$\mathrm{Aut}(C) := \{ f : C \to C$ isomorphism of curves$\}$

$\updownarrow$ 1:1

$\mathrm{Aut}(k(C)/k)$

Let C be a non-singular projective curve of genus $g(C) \geq 2$. Then

- $|\mathrm{Aut}(C)| < +\infty$ and

$$|\mathrm{Aut}(C)| \leq 84(g(C)-1)$$

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$\forall$ D of deg. 0 on C. $\exists!$ $P \in C$

such that $D \sim P - P_0$

apply RR to $D + P_0$

$$\ell(D+P_0) - \underbrace{\ell(K_C - D - P_0)}_{\deg < 0} = 1 + \deg(D+P_0) - 1$$

$$= 1$$

$\Rightarrow$ $\ell(D+P_0) = 1$ $\rightsquigarrow$ $\exists!$ effective divisor $P \sim_{\mathrm{eq}} D + P_0$

$P - P_0 \sim D$