Algebraic curves V

Hurwitz I Let $f: X \to Y$ be a separable morphism between two non-singular projective curves, then

$$2g(X)-2 \geq (\deg f)(2g(Y)-2)$$

If f has tame ramification everywhere

$$2g(X)-2 = (\deg f)(2g(Y)-2) + \sum_{P \in X}(e_P-1)$$

$h \in K(Y)$ $\qquad$ $2g(Y)-2 = \deg\big(\underbrace{\operatorname{div}(dh)}_{\in \operatorname{Div}(Y)}\big)$

$\downarrow$

$h^{\#} \in K(X)$ $\qquad$ $2g(X)-2 = \deg\big(\underbrace{\operatorname{div}(dh^{\#})}_{\in \operatorname{Div}(X)}\big)$

- f separable $\Rightarrow \operatorname{div}(dh^{\#}) \neq 0$

- $P \in X$ $\qquad$ $(t_P) = m_P$ $\qquad$ $t_Q = t_P^{e_P} \cdot u, \quad u \in \mathcal{O}_{X,P}^{*}$

$Q = f(P) \in Y$ $\qquad$ $\rightsquigarrow (t_Q) \in m_Q$ $\qquad$ tame ramification $\Rightarrow p \nmid e_P$

$$dt_Q = \left(e_P t_P^{e_P-1} u + t_P^{e_P} \frac{du}{dt}\right) dt_P$$

Rmk.

(1) If $f: X \to Y$ is purely inseparable.

$\rightsquigarrow$ X and Y are isomorphic to each other.

(typical example: $\mathbb{P}^1 \to \mathbb{P}^1$ char $k = p$
$t \mapsto t^p$)

(2). $\Longrightarrow$ Lüroth thm: Assume we have a dominant morphism.

$\mathbb{P}^1 \to C \hookleftarrow$ non-sing proj curve.

then $C \cong \mathbb{P}^1$.

Hurwitz 2

Char. $k = 0$

Assume. C non-singular projective curve of genus $g \geq 2$, Then $\underbrace{\mathrm{Aut}(C)}$ is finite;

and

$$\underbrace{|\mathrm{Aut}(C)|}_{\text{a finite group}} \leq 84(g-1)$$

Proof, assume $G \hookrightarrow \mathrm{Aut}(C) \rightsquigarrow G \subseteq \mathrm{Aut}\left(K(C)/k\right)$

$\rightsquigarrow$ let. $K(D) = K(C)^{G} \hookrightarrow K(C)$

quotient morphism

$\rightsquigarrow$ get a dominant morphism $\boxed{f: C \to D}$ between non-singular projective curves

$$\deg f = |G| =: n$$

apply Hurwitz 1: $2g(C)-2 = 2g-2 = n(2g(D)-2) + \sum_{P\in C}(e_P - 1)$

$\forall\, Q \in D$ $\quad f^{-1}Q = \{P_1, \dots, P_r\} \circlearrowleft G \leftarrow$ transitive.

P_2

P_1 $\quad z \mapsto z^a$

$e_{P_i} = e_{P_1} \quad \forall i.$ (f is Galois)

$f^{-1}Q = \{ P_1 \cdots P_{\frac{n}{r}} \} \rightsquigarrow e_{P_i} = r$ $\left(\deg f = \sum_{i=1}^{n/r} e_{P_i} \right)$

$\hookrightarrow \quad 2g-2 = n \cdot (2g(D)-2) + \sum_{Q \in D} \frac{n}{r_Q} (r_Q - 1)$

$\Rightarrow \quad n \left[2g(D) - 2 + \sum_{Q \in D} \left(1 - \frac{1}{r_Q} \right) \right] = 2g-2.$

then it suffices to bound below

$$2g(D)-2+\sum_{\theta\in D}\left(1-\frac{1}{r_\theta}\right) \quad \left(\ge \frac{1}{42}\right)$$

left ≥ 2.

$g(D) \ge 0$ ok.

$g(D) = 1$ left $\ge \frac{1}{2}$ ok.

$g(D) = 0$ left $= -2+\sum_i \left(1-\frac{1}{r_i}\right)$

$r_1 = 2$ $r_2 = 3$ $r_3 = 7$ $\Rightarrow$ get $\frac{1}{42}$.

Rmk

- Hurwitz's bound is optimal for ∞ many genus,

$g=3$. Klein quartic.

$$C := \{ X^3Y + Y^3Z + Z^3X = 0 \} \subseteq \mathbb{P}^2_k.$$

$\mathrm{Aut}(C)$ 168. simple group

- $\underline{\underline{\mathrm{Aut}(C)}}$.

genus	deg of canonical divisor.	Aut
$g=0$	$\rightsquigarrow 2g-2=-2<0$ (Fano variety)	$PGL_2(k)$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix}/\sim$
$g=1$	$\rightsquigarrow 0$ (Calabi-Yau).	$\mathrm{Aut}(C) \supseteq C$
$g\geq 2$	$-\ >0$ (general type).	finite group

$\underline{\dim 2}$ Xiao Gang: S non-sing proj surface "$K_S>0$"

$$|\mathrm{Aut}(S)| \leq 42^2 \cdot \left(\deg K_S^2 \text{ "}K_S^2\text{"}\right)$$

$\dim \geq 3$ Hacon-McKernan-Xu. X non-sing proj variety of dim n. "$K_X>0$"

then $|\mathrm{Aut}(X)| < C(n)\,(\deg K_X)$

$C(n)$ constant.

Embeddings of curves. the canonical embedding

C non-singular projective curve.

D divisor on C.

assume. $L(D) \neq 0$

$$L(D) = \{ f \in k(C) \mid \operatorname{div}(f) + D \geq 0 \}$$

$\downarrow$

linear space. $/k$.

$$|L(D)| = \{ \text{effective divisors linearly equivalent to } D \}$$

$$f \longmapsto \operatorname{div}(f) + D$$

$$\operatorname{div} f_1 = \operatorname{div} f_2 \Longleftrightarrow \operatorname{div}(f_1/f_2) = 0 \longleftrightarrow f_1 = c f_2,\ c \in k^*.$$

Given D, we can define a rational map

$$\varphi_D : C \dashrightarrow \mathbb{P}(L(D)^*) = \mathbb{P}^r.$$

$$P \longmapsto (f_0(P) : \cdots : f_r(P)).$$

$L(D) = \langle f_0, \ldots, f_r \rangle$

$\ell(D) = r+1$

- φ_D is a well-defined morphism

$$\Updownarrow$$

D is base-point-free.

$P \in C$ is called a base point of D

if $\forall \underbrace{\tilde{D}}_{\text{effective divisor}} \sim D$, $\quad \operatorname{Supp} \tilde{D} \ni P$.

If D has no base point, then we say D is base-point-free.

We say D is very ample,

If φ_D is a well-defined and φ_D gives an immersion from C to $\mathbb{P}_k^n$.

We say D is ample,

if $\exists\, N \in \mathbb{N}_{>0}$ such that $N \cdot D$ is very ample.

<u>Prop!</u> Let D be a divisor on C.

- P not a base-point of D iff

$$\ell(D-P) = \ell(D) - 1.$$

- D is very ample iff

$\forall P, Q \in C$, $$\ell(D-P-Q) = \ell(D) - 2.$$

(In general, $\ell(D-P) \ge \ell(D)-1$.)

Corollary 1: Assume $g(C)=g$.

Then
- $\forall D$, $\deg D \ge 2g$, D is base-point-free.
- $\forall D$, $\deg D \ge 2g+1$, D is very ample.

~~$\forall X$ proj variety~~
D ample divisor
$X \setminus D$ affine

$\forall\, U \subsetneq C$ open subset, U affine curve.

Corollary 2: Any nonzero effective divisor D on C is ample, Proof (1) If $\deg D \geq 2g$. $\deg K_C = 2g-2$

RR: $\ell(D) - \underbrace{\ell(K_C - D)}_{0} = 1+d-g$.

$\Rightarrow \ell(D) = 1+d-g$.

$\ell(D-P) - \underbrace{\ell(K_C-(D-P))}_{0} = 1+(d-1)-g$

$\Rightarrow \ell(D-P) = \ell(D) - 1 \quad \forall P \in C$.

D b-p-f.

(2) similar arguments shows $\ell(D-P-Q) = \ell(D)-2$,

Example $g=1$. assume C nonsing proj of genus 1.

$\forall P_0 \in C$. $3P_0$ very ample

$$L(0) \subseteq L(P_0) \subseteq L(2P_0) \subsetneq L(3P_0)$$

$L(0) = k$. $L(P_0) = k$. $L(2P_0)$: dim 2 $= \langle 1, x \rangle$, $x \in K(C) \setminus k$. $L(3P_0)$: dim 3 $= \langle 1, x, y \rangle$.

$\mathrm{ord}_{P_0} x = -2$, $\mathrm{ord}_{P_0} y = -3$.

$\ell(nP_0) = 1 + n - 1 = n$.

$\varphi_{3P_0} : C \hookrightarrow \mathbb{P}^2_k$. immersion.

$\mathrm{div}(x) + 2P_0 \geq 0$

$L(4P_0) = \langle 1,\ x,\ y,\ x^2 \rangle$

$\| \quad \dim = 4$

$L(5P_0) = \langle 1,\ x,\ y,\ x^2,\ xy \rangle$

$y^2 = (x-a)(x-b)(x-c)$

$L(6P_0) = \langle 1,\ x,\ y,\ x^2,\ xy,\ x^3,\ y^2 \rangle$

$\| \quad \dim = 6$

$\downarrow$ linearly dependent $/k$

$\leadsto \ y^2 + a_1 xy + a_2 y = x^3 + a_3 x^2 + a_4 x + a_5.$

the canonical embedding

K_C a canonical divisor of C

C, $g(C) \geq 2$.

$\deg K_C = 2g-2 > 0$.

$\ell(K_C) = g \geq 2$.

$\varphi_{K_C} : C \longrightarrow \mathbb{P}^{g-1}$ the canonical map of C.

- K_C base-point-free,

$\ell(K_C - P) = \ell(K_C) - 1 = g-1$?

$\ell(P) > 1 \Leftrightarrow C \simeq \mathbb{P}^1$

RR

$\ell(K_C - P) - \ell(P) = 1 + 2g-3 - g = g-2$

$\ell(P) \geq \ell(0) = k \Rightarrow \ell(K_C - P) = g-1$

$g(C)\geq 2$.

- C is called hyperelliptic if $\exists\ f: C \to \mathbb{P}^1$, $\deg f = 2$

Lemma. ($g(C)\geq 2$) K_C very ample $\iff$ C not hyperelliptic. (g^1_2 unique),

proof: $\forall P$.

$$\left.\begin{aligned} &\ell(K_C-P) = \ell(K_C)-1 = g-1,\\ &\ell(K_C-P-Q) = \ell(K_C)-2 = g-2. \end{aligned}\right\} \text{ by RR}$$

$\cdot \quad P, Q \in C \quad \text{s.t.} \quad \underbrace{\ell(K_C - P - Q)}_{(=g-1)} > \underbrace{\ell(K_C) - 2}_{=g-2}$

$\Longleftrightarrow \quad \underbrace{\ell(P+Q)}_{(=2)} > 1.$

$\Longleftrightarrow \quad \varphi_{P+Q} : C \to \mathbb{P}^1$ is a deg 2 morphism

<u>Rmk</u> · • genus 2 curves are always hyperelliptic

$\varphi_{K_C}: C \longrightarrow \mathbb{P}^1$

C is a

• char k = 0, genus g hyperelliptic curve.

$\exists f: C \xrightarrow{2:1} \mathbb{P}^1$

$$2g-2 = 2\times(-2) + \sum (\overset{2}{e_i} - 1)$$

f branched at 2g+2 points of $\mathbb{P}^1$

it has "moduli" 2g-1

Def. Assume $g \geq 3$, C non-hyperelliptic

then $\varphi_{K_C}: C \hookrightarrow \mathbb{P}^{g-1}$,

the image is called a canonical curve.

Example $g=3$, C non-hyperelliptic, $\varphi_{K_C}: C \hookrightarrow \mathbb{P}^2$

Example $g=4$ C non-hyperelliptic $\varphi_{K_C}: C \hookrightarrow \mathbb{P}^3$ deg 4 plane curve

$\rightsquigarrow$ $\exists$ quadric Q on $\mathbb{P}^3_k$ s.t. $C = Q \cap V$

cubic V on $\mathbb{P}^3_k$

<u>Rmk</u> In general $C \hookrightarrow \mathbb{P}^{g-1}$ canonical curve

- Max Noether: C projective normal $\rightarrow$ $\mathrm{Sym}^m L(K_C) \twoheadrightarrow L(mK_C)$ $\forall m \geq 1$
- <u>Petri</u>. $C \hookrightarrow \mathbb{P}^{g-1}$ ideals generated by quadrics (cubics)

Rmk

Deligne-Mumford $\Rightarrow$ irreducible variety.

$\exists\ M_g$. coarse moduli space of genus g non-singular proj curve.

$\cup\!\!\mid$

$Q \rightsquigarrow$ an isomorphic class of curves

Harris-Mumford: M_g complicated when $g \geq 25$.