2020 Algebraic Geometry Summer School
Exam B

1. (20 points) Let k be an algebraically closed field in characteristic 0. Let $P(t) = 2t + 2$.

1. Show that the Hilbert scheme $\mathrm{Hilb}(P, \mathbb{P}^3_k)$ has at least two irreducible components: one component H_0 of dimension 8 and one componnet H_1 of dimension 11.

2. Descibe the elements in $H_0 \cap H_1$.

Proof. As the Hilbert polynomial $P(t) = 2t + 2$ is of degree 1, any closed point of $\mathrm{Hilb}(P, \mathbb{P}^3_k)$ contains a degree 2 curve in $\mathbb{P}^3_k$. Then by the Castelnuovo inequality, we can see $g(C) = 0$ for any integral curve $[C] \in \mathrm{Hilb}(P, \mathbb{P}^3_k)$. There following two cases:

1. $P = (t + 1) + (t + 1)$. In this case, it is a union of two $\mathbb{P}^1_k$ in $\mathbb{P}^3_k$. Hence the dimension of this component is $2 \dim(\mathbb{P}^3_k)^* = 8$.

2. $P = (2t + 1) + 1$. In this case, it is a union of a conic and a point in $\mathbb{P}^3_k$. Hence the dimension is[1]
$$\dim((\mathbb{P}^3_k)^{[1]}) + \dim(\mathrm{Hilb}(2t + 1, \mathbb{P}^3_k)) = 11.$$

For the second question, you only need to note that the intersection of these two components contains the degenerated conics, which are two $\mathbb{P}^1_k$ that intersect at one point. □

2. (20 points) Let S be a smooth surface of degree d in $\mathbb{P}^3$ containing a line ℓ.

1. Compute ℓ^2.

2. Prove that the planes through ℓ cut out a pencil $|F|$ with $F^2 = 0$. Prove that $|F|$ is base point free, and defines a morphism $S \to |F|^\vee \cong \mathbb{P}^1$.

3. We suppose $d = 3$, and that S contains the lines $X = Y = 0$, $Z = T = 0$, $Y = T = 0$. Show that the rational map $\varphi : S \dashrightarrow \mathbb{P}^2$, $\varphi(X, Y, Z, T) = (XT, YT, YZ)$ extends to a morphism $S \to \mathbb{P}^2$ (use b) to extend φ along the 3 lines).

Proof.

1. We use adjunction formula to calculate the canonical divisor $\mathcal{O}_{\mathbb{P}^1}(-2)$ on L: $-2 = K_L|_L = (L + K_S)|_L = (L + O(d - 4))|_L = L^2 + d - 4$, therefore $L^2 = 2 - d$.

2. $d = (F + L)^2 = F^2 + 2FL + L^2$, note that by Bezout $FL = d - 1$, therefore $F^2 = d - 2d + 2 + d - 2 = 0$. Since F sweep through all points on L, it suffices to show F has no base points on L, but any base point contributes $+1$ to $F^2 = 0$. It defines a morphism to $\mathbb{P}^1$ as it is base point free.

3. Setting the homogeneous coordinates all be zero, we see the undefined locus are the union of three lines $X = Y = 0$ $T = Y = 0$ and $T = Z = 0$ in $\mathbb{P}^3$. Since S contains all these three (-1)-curves, it suffices to show ϕ extends to a morphism on each of them. We just explicitly write them down by cancelling the equal factors: For example on $(X = Y = 0)$, we extend it as $(X, Y, Z, T) \to (T, T, Z)$, it is well defined because by $b)$ $(X, Y, Z, T) \to (T, Z)$ is well defined, and $\mathbb{P}^1 \to \mathbb{P}^2$: $(A, B) \mapsto (A, A, B)$ is well-defined.

□

[1] There is a typo in the test that the "dimension 10 component" should be "dimension 11 component"

3. (20 points) Let $k \geq 2$ be an integer. Recall that we have the Eisenstein series

$$G_{2k}(\tau) = \sum_{(c,d)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(c\tau+d)^{2k}},$$

which is an element in $\mathcal{M}_{2k}(\mathrm{SL}_2(\mathbb{Z}))$.

1. Show that for every prime p, G_{2k} is an eigenvector for the Hecke operator T_p with eigenvalue $\sigma_{2k-1}(p)$.
2. Show that for every integer $n \geq 1$, G_{2k} is an eigenvector for the Hecke operator T_n with eigenvalue $\sigma_{2k-1}(n)$.
3. Let F_{2k} be the normalized Hecke eigenform that is proportional to G_{2k}. Use (2) to conclude that
$$F_{2k}(\tau) = c_{2k} + \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n$$
for some constant c_{2k}.
4. For a normalized Hecke eigenform $f(\tau) = \sum_{n=0}^{\infty} a_n(f) q^n$, we define its L-function to be
$$L(s, f) = \sum_{n=1}^{\infty} a_n(f) n^{-s},$$
where s is a complex variable. Express $L(s, F_{2k})$ in terms of the Riemann zeta function $\zeta(s)$. (You may ignore the issue of convergence.)

Proof. 1. By definition of T_p:

$$\begin{aligned}
T_p G_{2k}(\tau) &= \frac{1}{p}\sum_{j=0}^{p-1} G_{2k}(\frac{\tau+j}{p}) + p^{2k-1} G_{2k}(p\tau) \\
&= \frac{1}{p}\sum_{j}\sum_{c,d} \frac{1}{(c\frac{\tau+j}{p}+d)^{2k}} + p^{2k-1}\sum_{c,d}\frac{1}{(cp\tau+d)^{2k}} \\
&= A + B
\end{aligned}$$

where A denotes the first sum and B the second. We have:

$$\begin{aligned}
A &= \frac{1}{p}\sum_{p\nmid c}\sum_{j}\sum_{d} \frac{p^{2k}}{(c\tau+cj+dp)^{2k}} + \frac{1}{p}\sum_{c=pc'}\sum_{j}\sum_{d}\frac{1}{(c'\tau+c'j+d)^{2k}} \\
&= C + G_{2k}(\tau)
\end{aligned}$$

where C denotes the sum $\frac{1}{p}\sum_{p\nmid c}\sum_j\sum_d \frac{p^{2k}}{(c\tau+cj+dp)^{2k}}$. We have:

$$\begin{aligned}
C &= p^{2k-1}\sum_{p\nmid c}\sum_{d'\in\mathbb{Z}} \frac{1}{(c\tau+d')^{2k}} \\
&= p^{2k-1}(\sum_{c,d}\frac{1}{(c\tau+d)^{2k}} - \sum_{p|c}\sum_{d}\frac{1}{(c\tau+d)^{2k}}) \\
&= p^{2k-1}G_{2k}(\tau) - B
\end{aligned}$$

Therefore,

$$\begin{aligned} T_pG_{2k}(\tau) &= C + B + G_{2k}(\tau) \\ &= (p^{2k-1}+1)G_{2k}(\tau) \\ &= \sigma_{2k-1}(p)G_{2k}(\tau) \end{aligned}$$

2. One proves first inductively $T_{p^r}G_{2k} = \sigma_{2k-1}(p^r)G_{2k}$. Suppose it's true for $\leq r-1$, Then

$$\begin{aligned} T_{p^r}G_{2k} &= T_pT_{p^{r-1}}G_{2k} - p^{2k-1}T_{p^{r-2}}G_{2k} \\ &= T_p\sigma_{2k-1}(p^{r-1})G_{2k} - p^{2k-1}\sigma_{2k-1}(p^{r-2})G_{2k} \\ &= (\sigma_{2k-1}(p)\sigma_{2k-1}(p^{r-1}) - p^{2k-1}\sigma_{2k-1}(p^{r-2}))G_{2k} \\ &= \sigma_{2k-1}(p^r)G_{2k} \end{aligned}$$

Then, for $n = p_1^{e_1}\cdots p_s^{e_s}$, we have $T_nG_{2k} = T_{p_1^{e_1}}\cdots T_{p_s^{e_s}}G_{2k} = \sigma_{2k-1}(p_1^{e_1})\cdots\sigma_{2k-1}(p_s^{e_s})G_{2k} = \sigma_{2k-1}(n)G_{2k}$, by direct verification.

3. One proves the following statement, and apply (2):

If $f = \sum_{n=0}^{\infty} a_nq^n$ is a normalized eigenform, then $a_1(T_nf) = a_n$ for $n \in \mathbb{N}^+$

Proof. For prime p, we have $a_1(T_pf) = a_p + p^{2k-1}a_{\frac{1}{p}} = a_p$. We prove by induction that $a_1(T_{p^r}f) = a_{p^r}$:

$$\begin{aligned} a_1(T_{p^r}f) &= a_1(T_{p^{r-1}}T_rf) - p^{2k-1}a_1(T_{p^{r-2}}f) \\ &= a_{p^{r-1}}(T_rf) - p^{2k-1}a_{p^{r-2}} \\ &= a_{p^r} + p^{2k-1}a_{p^{r-2}} - p^{2k-1}a_{p^{r-2}} \\ &= a_{p^r} \end{aligned}$$

for $r \nmid n$,suppose $a_1(T_nf) = a_n$, then,

$$\begin{aligned} a_1(T_{np^r}f) &= a_1(T_nT_{p^r}f) = a_n \\ &= a_n(a_{p^r}f) = a_na_{p^r} \end{aligned}$$

Hence it's also true for np^r. □

4.

$$\begin{aligned} L(s, F_{2k}) &= \sum_{n=1}^{\infty} \frac{\sigma_{2k-1}(n)}{n^s} = \prod_p \sum_{r=0}^{\infty} \frac{\sigma_{2k-1}(p^r)}{p^{rs}} \\ &= \prod_p \sum_r \frac{1 + p^{2k-1} + \cdots + p^{(2k-1)r}}{p^{rs}} \\ &= \prod_p \sum_r \frac{1}{p^{2k-1}-1}(p^{(2k-1)r}p^{2k-1} - p^{-rs}) \\ &= \prod_p \frac{1}{(1-p^{-s})(1-p^{2k-1-s})} \\ &= \zeta(s)\zeta(s-2k+1) \end{aligned}$$

□

4. (20 points) Let C be a non-hyperelliptic genus g curve, Let

$$\varphi_{\omega_C} = \varphi_{|\omega_C|} : C \hookrightarrow \mathbb{P}(H^0(C, \omega_C)) = \mathbb{P}^{g-1}$$

1. Suppose C is trigonal. So then there is an effective divisor D of degree 3 such that $h^0(\mathcal{O}_C(D)) = 2$. Show that $\varphi_{\omega_C}(D)$ gives you 3 collinear points in $\mathbb{P}^{g-1}$. Show that $I_{C/\mathbb{P}^{g-1}}$ can not be generated by quadrics.

2. Assume C is a smooth plane curve of degree 5, $\omega_C = \mathcal{O}_{\mathbb{P}^2}(2)|_C$. Let

$$\nu_2 = \varphi_{\mathcal{O}_{\mathbb{P}^2}(2)} : \mathbb{P}^2 \to \mathbb{P}^5.$$

Then $\nu_2(\mathbb{P}^2)$ is a degree 4 surface in $\mathbb{P}^5$. Show that every quadric hypersurface containing C will also containing S.

Proof. 1. By Riemann Roch on curve, we have

$$l(D) - l(K - D) = \deg\, D + 1 - g = 4 - g$$

Hence $l(K-D) = g-2$, and that means D is collinear in the canonical embedding, otherwise $l(K - D) = g - 3$. Assume Q is a quadric containing C, then, the 3 collinear points are contained in Q. Since degree $Q = 2$, the line L generated by the 3 point is contained in Q. If $I_{C/\mathbb{P}^{g-1}}$ are generated by quadrics, then we have $L \subset C$, which is impossible.

2. Let Q be a quadric not containing S, then $Q \cap S$ is a curve of degree $2 \times 4 = 8$ in $\mathbb{P}^5$ by Bezout. But $C \subset Q \cap S$ already has degree $2 \times 5 = 10 > 8$, therefore C is not contained in $Q \cap S$, not contained in Q.

□