COMMUTATIVE ALGEBRA NOTES

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Contents

1. Introduction

In this lecture, we consider a (Noetherian) commutative ring R with identity element.

I will assume that students know about basic definitions and properties of rings, ideals, modules, morphisms (e.g. Chapter 1–3 of [1]). Our main textbook is [2]. We will cover selected topics in order to serve the lecture of geometry of syzygies ([2, Section 17–19]).

1.1. **Nakayama's lemma.**

The *Jacobson radical* $J(R)$ of R is the intersection of all maximal ideals. Note that $y \in J(R)$ iff $1 - xy$ is a unit in R for every $x \in R$.

Date: July 24, 2020.

Theorem 1.1 (Nakayama's lemma)**.** *Let I be an ideal contained in the Jacobson radical of R, and M a finitely generated R-module. If $IM = M$, then $M = 0$.*

Lemma 1.2. *Let I be an R-ideal and M a finitely generated R-module. If $IM = M$, then there exists $y \in I$ such that $(1 - y)M = 0$.*

Proof. This is a consequence of the Caylay–Hamilton theorem. Consider $m_1, \dots, m_n$ a set of generators in M, then there exists an $n \times n$ matrix A with coefficients in I such that $(m_1, \dots, m_n)^T = A(m_1, \dots, m_n)^T$. Set $\mathbf{m} = (m_1, \dots, m_n)^T$. Hence $(I_n - A)\mathbf{m} = 0$. Note that $\mathrm{adj}(I_n - A)(I_n - A) = \det(I_n - A)I_n$, we know that $\det(I_n - A)\mathbf{m} = 0$, that is, $\det(I_n - A)m_i = 0$ for all i. This implies that $\det(I_n - A)M = 0$. □

Example 1.3. If we do not assume that M is finitely generated, this is not true. For example, consider $R = k[[x]]$, $M = k[[x, x^{-1}]]$.

Corollary 1.4. *Let I be an ideal contained in the Jacobson radical of R, and M a finitely generated R-module. If $N + IM = M$ for some submodule $N \subset M$, then $M = N$.*

Proof. Apply Nakayama's lemma to M/N. □

Corollary 1.5. *Let $(R, \mathfrak{m})$ be a local ring and M a finitely generated R-module. Consider $m_1, \dots, m_n \in M$. If $\bar{m}_1, \dots, \bar{m}_n \in M/\mathfrak{m}M$ is a basis (as a $R/\mathfrak{m}$-vector space), then $m_1, \dots, m_n$ generates M (which is also a minimal set of generators.)*

Proof. Apply Corollary 1.4 to N the submodule generated by $m_1, \dots, m_n$. □

1.2. **Noetherian rings.**

Definition 1.6 (Noetherian ring)**.** A ring R is *Noetherian* if one of the following equivalent conditions holds:

(1) Every non-empty set of ideals has a maximal element;
(2) The set of ideals satisfies the ascending chain condition (ACC);
(3) Every ideal is finitely generated.

In this lecture, we assume all rings are Noetherian and all modules are finitely generated for simplicity.

Theorem 1.7 (Hilbert basis theorem)**.** *If R is Noetherian, then $R[x]$ is Noetherian.*

Idea of proof. Consider $I \subset R[x]$ an ideal. Consider $J \subset R$ the leading coefficients of I, then J is finitely generated. We may assume that J is generated by the leading coefficients of $f_1, \dots, f_n \in R[x]$. Take I' be the ideal generated by $f_1, \dots, f_n$, then it is easy to see that any $f \in I$ can be written as $f = f' + g$ with $f' \in I'$ and $\deg g < \max_i\{\deg f_i\} = r$. So

$$I = I \cap (R \oplus Rx \oplus \cdots \oplus Rx^{r-1}) + I'$$

is finitely generated. (Check that $I \cap (R \oplus Rx \oplus \cdots \oplus Rx^{r-1})$ is finitely generated!) □

Example 1.8. Any quotient of polynomial ring $k[x_1, \dots, x_n]/I$ is Noetherian.

1.3. **Associated primes.** We will use the notion $(A : B)$ to define the set $\{a \mid aB \subset A\}$ whenever it makes sense. For example, if $N, N' \subset M$ are R-modules and I an ideal, then we can define $(N : I)$ as a submodule of M, and $(N' : N)$ an ideal. Usually the set $(0 : N)$ is denoted by $\operatorname{ann}(N)$ and called the *annihilator* of N, that is, the set of elements whose multiplication action kills N.

Definition 1.9 (Associated prime)**.** A prime P of R is *associated* to M if $P = \operatorname{ann}(x)$ for some $x \in M$.

Associated primes are important in the primary decomposition. But here we mainly focus on its relation with zero-divisors.

Theorem 1.10. *Let R be a Noetherian ring and M a finitely generated R-module. Then the union of associated primes to M consists of zero and zero-divisors. Moreover, there are only finitely many associated primes.*

Proof. We want to show that

$$\bigcup_{\operatorname{ann}(x):\text{prime}} \operatorname{ann}(x) = \bigcup_{x \neq 0} \operatorname{ann}(x).$$

So it suffices to show that if $\operatorname{ann}(y)$ is maximal among all $\operatorname{ann}(x)$, then $\operatorname{ann}(y)$ is prime. Consider $rs \in \operatorname{ann}(y)$ such that $s \notin \operatorname{ann}(y)$, then $rsy = 0$ but $sy \neq 0$. We know that $\operatorname{ann}(y) \subset \operatorname{ann}(sy)$, so equality holds by maximality. This implies that $r \in \operatorname{ann}(y)$.

To prove the finiteness, we only outline the idea here. Denote $\operatorname{Ass}(M)$ the set of associated primes. Then it is not hard to see that for a short exact sequence

$$0 \to M' \to M \to M'' \to 0,$$

we have

$$\operatorname{Ass}(M') \subset \operatorname{Ass}(M) \subset \operatorname{Ass}(M') \cup \operatorname{Ass}(M'').$$

So inductively we get the finiteness. □

Remark 1.11. Another fact is that if P is a prime minimal among all primes containing $\operatorname{ann}(M)$, then P is an associated prime.

Corollary 1.12. *Let R be a Noetherian ring and M a finitely generated R-module. Let I be an ideal. Then either I contains a non zero-divisor on M, or I annihilated a non-zero element of M.*

Proof. Suppose that I contains only zero-divisors on M, then by Theorem 1.10, $I \subset \bigcup_{\operatorname{ann}(x):\text{prime}} \operatorname{ann}(x)$. So the conclusion follows from the following easy lemma. □

Lemma 1.13. *Let I be an ideal and let $P_1, \ldots, P_n$ be primes of R. If $I \subset \bigcup_i P_i$, then $I \subset P_i$ for some i.*

1.4. **Tensor products and Tor.** Let M, N be R-modules, the *tensor product* $M \otimes N$ is defined by the module generated by

$$\{m \otimes n \mid m \in M, n \in N\},$$

modulo relations

$$(m + m') \otimes n = m \otimes n + m' \otimes n;$$

$$m \otimes (n + n') = m \otimes n + m \otimes n';$$
$$(rm) \otimes n = m \otimes (rn) = r(m \otimes n)$$

for $m \in M, n \in N, r \in R$. It can be characterized by the universal property that if $f : M \times N \to P$ is an R-bilinear map, then there exists a unique $g : M \otimes N \to P$ such that f factors through g.

Example 1.14. (1) $M \otimes R \simeq M$, $M \otimes R^n \simeq M^n$;
(2) $M \otimes R/I \simeq M/IM$;
(3) $(M \otimes_R N)_P \simeq M_P \otimes_{R_P} N_P$.

Proposition 1.15. *$(- \otimes N)$ is a right-exact functor. If*

$$M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

is a exact sequence of R-modules, then

$$M' \otimes N \xrightarrow{f \otimes 1} M \otimes N \xrightarrow{g \otimes 1} M'' \otimes N \to 0$$

is exact.

Definition 1.16 (Flat module)**.** N is *flat* if $(- \otimes N)$ is an exact functor, that is, if

$$0 \to M' \to M \to M'' \to 0$$

is a exact sequence of R-modules, then

$$0 \to M' \otimes N \to M \otimes N \to M'' \otimes N \to 0$$

is exact.

To study flatness, we need to introduce Tor from homological algebra.

Definition 1.17 (Projective module)**.** An R-module M is *projective* if for any surjective map $f : N_1 \to N_2$ and any map $g : M \to N_2$, there exists $h : M \to N_1$ such that $f \circ h = g$.

Example 1.18. Free modules are flat and projective.

Definition 1.19 (Complexes and homologies)**.** A *complex* of R-modules is a sequence of R-modules with (differential) homomorphisms

$$\mathcal{F} : \cdots \to F_{i+1} \xrightarrow{\delta_{i+1}} F_i \xrightarrow{\delta_i} F_{i-1} \to \ldots$$

such that $\delta_i \delta_{i+1} = 0$ for each i. Denote the *homology* to be $H_i(\mathcal{F}) = \ker(\delta_i)/\mathrm{im}(\delta_{i+1})$. We say that $\mathcal{F}$ is *exact* at degree i if $H_i(\mathcal{F}) = 0$. A morphism of complexes $\phi : \mathcal{F} \to \mathcal{G}$ is given by $\phi_i : F_i \to G_i$ commuting with differentials, that is, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{F}: & \ldots \arrow[r] & F_{i+1} \arrow[r] \arrow[d, "\phi_{i+1}"] & F_i \arrow[r] \arrow[d, "\phi_i"] & F_{i-1} \arrow[r] \arrow[d, "\phi_{i-1}"] & \ldots \\
\mathcal{G}: & \ldots \arrow[r] & G_{i+1} \arrow[r] & G_i \arrow[r] & G_{i-1} \arrow[r] & \ldots
\end{tikzcd}$$

This naturally gives morphisms between homologies $\phi_i : H_i(\mathcal{F}) \to H_i(\mathcal{G})$.

Definition 1.20 (Projective resolution)**.** A *projective resolution* of an R-module M is a complex of projective modules

$$\mathcal{F} : \cdots \to F_n \to \cdots \to F_1 \xrightarrow{\phi_1} F_0$$

which is exact and $\operatorname{coker}(\phi_1) = M$. Sometimes we also denote it by

$$\mathcal{F} : \cdots \to F_n \to \cdots \to F_1 \xrightarrow{\phi_1} F_0 (\to M \to 0).$$

Definition 1.21 (Left derived functor)**.** Let T be a right-exact functor. Given a projective resolution of an R-module M:

$$\mathcal{F} : \cdots \to F_n \to \cdots \to F_1 \xrightarrow{\phi_1} F_0 (\to M \to 0).$$

Define the *left derived functor* by $L_i T(M) := H_i(T\mathcal{F})$, which is just the homology of

$$T\mathcal{F} : \cdots \to T(F_n) \to \cdots \to T(F_1) \to T(F_0) (\to T(M) \to 0).$$

We collect basic properties of derived functors here.

Proposition 1.22. (1) *$L_0 T(M) = T(M)$;*

(2) *$L_i T(M)$ is independent of the choice of projective resolution;*
(3) *If M is projective, then $L_i T(M) = 0$ for $i > 0$.*
(4) *For a short exact sequence of R-modules*

$$0 \to A \to B \to C \to 0,$$

we have a long exact sequence

$$\begin{aligned}
&\cdots \\
&\to L_3T(A) \to L_3T(B) \to L_3T(C) \\
&\to L_2T(A) \to L_2T(B) \to L_2T(C) \\
&\to L_1T(A) \to L_1T(B) \to L_1T(C) \\
&\to T(A) \to T(B) \to T(C) \to 0.
\end{aligned}$$

Definition 1.23 (Tor)**.** For an R-module N, $\operatorname{Tor}_i^R(-, N)$ is defined by $L_i T(-)$ where $T = (- \otimes N)$.

Remark 1.24. So to compute $\operatorname{Tor}_i^R(M, N)$, we should pick a projective resolution $\mathcal{F}$ of M and compute $H_i(\mathcal{F} \otimes N)$. Note that tensor products are symmetric, that is, $M \otimes N \simeq N \otimes M$, it can be seen that $\operatorname{Tor}_i^R(M, N) \simeq \operatorname{Tor}_i^R(N, M)$, and $\operatorname{Tor}_i^R(M, N)$ can be also computed by pick a projective resolution $\mathcal{G}$ of N and compute $H_i(M \otimes \mathcal{G})$.

Theorem 1.25. *TFAE:*

(1) *N is flat;*
(2) *$\operatorname{Tor}_i^R(M, N) = 0$ for all $i > 0$ and all M;*
(3) *$\operatorname{Tor}_1^R(M, N) = 0$ for all M.*

Proof. (1) $\implies$ (2): take a projective resolution $\mathcal{F}$ of M, we need to compute $H_i(\mathcal{F} \otimes N)$. As N is flat, $\mathcal{F} \otimes N$ is exact, hence $\operatorname{Tor}_i^R(M, N) = 0$ for all $i > 0$.

(2) $\implies$ (3): trivial.

(3) $\implies$ (1): this follows from the long exact sequence

$$\mathrm{Tor}_1^R(M'', N) \to M' \otimes N \to M \otimes N \to M'' \otimes N \to 0.$$

□

2. Koszul complexes and regular sequences

2.1. **Regular sequences.**

Definition 2.1 (Regular sequence)**.** Let R be a ring and M an R-module. A sequence of elements $x_1, \dots, x_n \in R$ is called a *regular sequence* on M (or *M-sequence*) if

(1) $(x_1, \dots, x_n)M \neq M$;
(2) For each $1 \leq i \leq n$, x_i is not a zero-divisor on $M/(x_1, \dots, x_{i-1})M$.

Definition 2.2 (Depth)**.** Let R be a ring, I an ideal, and M an R-module. Suppose $IM \neq M$. The *depth* of I on M, $\mathrm{depth}(I, M)$, is defined by the maximal length of M-sequences in I.

Remark 2.3. (1) If $M = R$, then simply denote $\mathrm{depth}\, I := \mathrm{depth}(I, M)$.
(2) We will see soon (Theorem 2.15) that any maximal M-sequence has the same length.

Example 2.4. If $R = k[x_1, \dots, x_n]$, then $x_1, \dots, x_n$ is a regular sequence. We will see soon that $\mathrm{depth}(x_1, \dots, x_n) = n$.

Remark 2.5. The depth measures the size of an ideal, and an element in the regular sequence corresponds to a hypersurface in geometry. So a regular sequence in I corresponds to a set of hypersurface containing $V(I)$ intersecting each other "properly". Consider for example $R = k[x, y]$ or $k[x, y]/(xy)$, $I = (x, y)$.

2.2. **Koszul complexes.**

Definition 2.6 (Complexes and homologies)**.** A *complex* of R-modules is a sequence of R-modules with homomorphisms

$$\mathcal{F} : \cdots \to M_{i-1} \xrightarrow{\delta_{i-1}} M_i \xrightarrow{\delta_i} M_{i+1} \to \dots$$

such that $\delta_i \delta_{i-1} = 0$ for each i. Denote the *(co)homology* to be $H^i(\mathcal{F}) = \ker(\delta_i)/\mathrm{im}(\delta_{i-1})$.

We will introduce Koszul complexes and explain how regular sequences are related to Koszul complexes.

Example 2.7 (Koszul complex of length 1)**.** Given $x \in R$. The Koszul complex of length 1 is given by

$$K(x) : 0 \to R \xrightarrow{x} R \to 0.$$

Note that $H^0(K(x)) = (0 : x)$, $H^1(K(x)) = R/xR$. Then x is an R-sequence if (1) $H^1(K(x)) \neq 0$; (2) $H^0(K(x)) = 0$.

Example 2.8 (Koszul complex of length 2)**.** Given $x, y \in R$. The Koszul complex of length 2 is given by

$$K(x,y) : 0 \to R \xrightarrow{\begin{pmatrix} y \\ x \end{pmatrix}} R^{\oplus 2} \xrightarrow{\begin{pmatrix} -x & y \end{pmatrix}} R \to 0.$$

Note that $H^0(K(x,y)) = (0 : (x,y))$. $H^2(K(x,y)) = R/(x,y)R$. We can compute $H^1(K(x,y))$ (Exercise). It turns out that if x is not a zero-divisor in R, then $H^1(K(x,y)) \simeq (x : y)/(x)$. So $H^1(K(x,y)) = 0$ if and only if y is not a zero-divisor of $R/(x)$. In conclusion, x, y is an R-sequence if (1) $H^2(K(x,y)) \neq 0$; (2) $H^0(K(x,y)) = H^1(K(x,y)) = 0$.

Theorem 2.9. *Let $(R, \mathfrak{m})$ be a local ring and $x, y \in \mathfrak{m}$. Then x, y is a regular sequence iff $H^1(K(x,y)) = 0$. In particular, x, y is a regular sequence iff y, x is a regular sequence.*

Proof. This is not a direct consequence of the above argument, as we need to show that x is a non-zero-divisor (equivalent to $H^0(K(x)) = 0$). Write $K(x,y)$ as the following:

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "x"] \arrow[dr, "y"] & R \arrow[r] \arrow[dr, "y"] \arrow[d, phantom, "\oplus"] & 0 & \\
& 0 \arrow[r] & R \arrow[r, "-x"] & R \arrow[r] & 0.
\end{tikzcd}$$

Then this gives a short exact sequence of complexes

$$\begin{tikzcd}
K(x)[-1] : & 0 \arrow[r] \arrow[d] & R \arrow[r, "-x"] \arrow[d, "i_2"] & R \arrow[r] \arrow[d, "1"] & 0 . \\
K(x,y) : 0 \arrow[r] & R \arrow[r] \arrow[d, "1"] & R^2 \arrow[r] \arrow[d, "p_1"] & R \arrow[r] \arrow[d] & 0 \\
K(x) : 0 \arrow[r] & R \arrow[r, "x"] & R \arrow[r] & 0 &
\end{tikzcd}$$

That is,

$$0 \to K(x)[-1] \to K(x,y) \to K(x) \to 0.$$

Then this induces a long exact sequences of homologies

$$H^0(K(x)) \xrightarrow{y} H^0(K(x)) \to H^1(K(x,y)) \to H^1(K(x)).$$

So $H^1(K(x,y)) = 0$ implies that $yH^0(K(x)) = H^0(K(x))$, which means that $H^0(K(x)) = 0$ by Nakayama's lemma. $\square$

Corollary 2.10. *Let $(R, \mathfrak{m})$ be a local ring and $x_1, \ldots, x_n \in \mathfrak{m}$. Suppose that $x_1, \ldots, x_n$ is a regular sequence, then any permutation of $x_1, \ldots, x_n$ is again a regular sequence. (Exercise.)*

We will define Koszul complexes and show this correspondence in general.

Definition 2.11 (Exterior algebra)**.** Let N be an R-module. Denote the *tensor algebra*

$$T(N) = R \oplus N \oplus (N \otimes N) \oplus \ldots$$

The *exterior algebra* $\bigwedge N = \oplus_m \bigwedge^m N$ is defined by $T(N)$ modulo the relations $x \otimes x$ (and hence $x\otimes y + y \otimes x$) for $x, y \in N$. The product of $a, b \in \bigwedge N$ is written as $a \wedge b$.

Definition 2.12 (Koszul complex)**.** Let N be an R-module, $x \in N$. Define the *Koszul complex* to be

$$K(x) : 0 \to R \to N \to \bigwedge^2 N \to \cdots \to \bigwedge^i N \xrightarrow{d_x} \bigwedge^{i+1} N \to \dots$$

where d_x sends a to $x \wedge a$. If $N \simeq R^n$ is a free module of rank n (we always consider this situation) and $x = (x_1, \dots, x_n) \in R^n$, then we denote $K(x)$ by $K(x_1, \dots, x_n)$.

Remark 2.13.
(1) The $R \to N$ maps 1 to x.
(2) Consider $N = R^2$ (with basis e_1, e_2) and $x = (x_1, x_2)$, then $\bigwedge^2 N \simeq R$ (with bases $e_1 \wedge e_2$), and the map $N \to \bigwedge^2 N$ is given by $e_1 \mapsto (x_1e_1 + x_2e_2) \wedge e_1 = -x_2 e_1 \wedge e_2$ and $e_2 \mapsto x_1 e_1 \wedge e_2$. In other words,

$$K(x_1, x_2) : 0 \to R \xrightarrow{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} R^{\oplus 2} \xrightarrow{\begin{pmatrix} -x_2 & x_1 \end{pmatrix}} R \to 0.$$

Example 2.14. $H^n(K(x_1, \dots, x_n)) = R/(x_1, \dots, x_n)$. Consider the corresponding complex

$$\bigwedge^{n-1} N \xrightarrow{d_x} \bigwedge^n N \to \bigwedge^{n+1} N = 0$$

Denote $e_1, \dots, e_n$ to be a basis of $N \simeq R^n$, then the basis of $\bigwedge^n N$ is just $e_1 \wedge \cdots \wedge e_n$, and the basis of $\bigwedge^{n-1} N$ is $e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_n$ $(1 \leq i \leq n)$. d_x maps $e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_n$ to $(-1)^{i-1} x_i e_1 \wedge \cdots \wedge e_n$. So $\mathrm{im} d_x = (x_1, \dots, x_n)$ and $H^n(K(x_1, \dots, x_n)) = R/(x_1, \dots, x_n)$.

2.3. **Koszul complexes versus regular sequences.** Now we can state the main theorem of this section.

Theorem 2.15. *Let M be a finitely generated R-module. If*

$$H^j(M \otimes K(x_1, \dots, x_n)) = 0$$

for $j < r$ and $H^r(M \otimes K(x_1, \dots, x_n)) \neq 0$, then every maximal M-sequence in $I = (x_1, \dots, x_n) \subset R$ has length r.

Idea of proof. Firstly, we consider the case that $x_1, \dots, x_s$ is a maximal M-sequence. In this case it is natural to prove this case by induction on n and s.

In order to reduce the general case to this case, we consider $y_1, \dots, y_s$ a maximal M-sequence, and consider $H^j(M \otimes K(y_1, \dots, y_s, x_1, \dots, x_n))$.

So to deal with both cases, we need to investigate the relation between $K(y_1, \dots, y_s, x_1, \dots, x_n)$ and $K(x_1, \dots, x_n)$ and the relation of their homologies. $\square$

Corollary 2.16. *If $x_1, \dots, x_n$ is an M-sequence, then $H^j(M \otimes K(x_1, \dots, x_n)) = 0$ for $j < n$ and $H^n(M \otimes K(x_1, \dots, x_n)) = M/(x_1, \dots, x_n)M$.*

Proof. By definition, $\operatorname{depth}(I, M) \geq n$, so $H^j(M \otimes K(x_1, \ldots, x_n)) = 0$ for $j < n$. On the other hand,

$$\begin{aligned} H^n(M \otimes K(x_1, \ldots, x_n)) &= \operatorname{coker}(M \otimes \bigwedge^{n-1} N \to M \otimes \bigwedge^n N) \\ &= M \otimes \operatorname{coker}(\bigwedge^{n-1} N \to \bigwedge^n N) \\ &= M \otimes R/(x_1, \ldots, x_n) = M/(x_1, \ldots, x_n)M. \end{aligned}$$

Here we use the fact that $M \otimes -$ is right-exact. □

Theorem 2.15 can be strengthen for local rings.

Theorem 2.17. *Let $(R, \mathfrak{m})$ be a local ring, $x_1, \ldots, x_n \in \mathfrak{m}$. Let M be a finitely generated R-module. If $H^k(M \otimes K(x_1, \ldots, x_n)) = 0$ for some k, then $H^j(M \otimes K(x_1, \ldots, x_n)) = 0$ for all $j < r$. Moreover, if $H^{n-1}(M \otimes K(x_1, \ldots, x_n)) = 0$, then $x_1, \ldots, x_n$ is an M-sequence.*

Corollary 2.18. *If R is local and $(x_1, \ldots, x_n)$ is a proper ideal containing an M-sequence of length n, then $x_1, \ldots, x_n$ is an M-sequence.*

Proof. $H^n(M \otimes K(x_1, \ldots, x_n)) = M/(x_1, \ldots, x_n)M \neq 0$ by Nakayama's lemma. Take r minimal such that $H^r(M \otimes K(x_1, \ldots, x_n)) \neq 0$, then every maximal M-sequence in $(x_1, \ldots, x_n)$ has length r, which implies that $r \geq n$. So $H^{n-1}(M \otimes K(x_1, \ldots, x_n)) = 0$ and $x_1, \ldots, x_n$ is an M-sequence. □

2.4. **Operations on Koszul complexes.**

Definition 2.19 (Tensor product of two complexes)**.** Given two complexes

$$\mathcal{F} : \cdots \to F_i \xrightarrow{\phi_i} F_{i+1} \to \ldots;$$

$$\mathcal{G} : \cdots \to G_i \xrightarrow{\psi_i} G_{i+1} \to \ldots$$

define the tensor product

$$\mathcal{F} \otimes \mathcal{G} : \cdots \to \bigoplus_{i+j=k} F_i \otimes G_j \xrightarrow{d_k} \bigoplus_{i+j=k+1} F_i \otimes G_j \to \ldots,$$

where the map $F_i \otimes G_j \to F_{i'} \otimes G_{j'}$ is $\begin{cases} \phi_i \otimes 1 & \text{if } i' = i+1; \\ (-1)^i 1 \otimes \psi_j & \text{if } j' = j+1; \\ 0 & \text{otherwise.} \end{cases}$ (Check $dd = 0$.)

Definition 2.20 (Shift)**.** Given a complex

$$\mathcal{F} : \cdots \to F_i \xrightarrow{\phi_i} F_{i+1} \to \ldots;$$

Denote $\mathcal{F}[n]$ to be the complex obtained by shifting $\mathcal{F}$ (to the left) n times. That is, $\mathcal{F}[n]_i = \mathcal{F}_{n+i}$, and the differential is multiplied by $(-1)^n$. Denote $R[n]$ to be the simple complex whose n-th position is R. Note that $\mathcal{F}[n] = R[n] \otimes \mathcal{F}$.

Definition 2.21 (Mapping cone)**.** For $y \in R$, consider $\mathcal{F} = K(y)$, that is,

$$\mathcal{F} : 0 \to R \xrightarrow{y} R \to 0.$$

Then there is a natural exact sequence of complexes

$$0 \to R[-1] \to \mathcal{F} \to R \to 0.$$

Tensoring a complex $\mathcal{G}$, this gives an exact sequence

$$0 \to \mathcal{G}[-1] \to \mathcal{F} \otimes \mathcal{G} \to \mathcal{G} \to 0.$$

Here $\mathcal{F} \otimes \mathcal{G}$ is the *mapping cone* of the map $\mathcal{G} \xrightarrow{y} \mathcal{G}$, in fact, it is given by

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[dr] & G_i \arrow[r, "(-1)^i\psi_i"] \arrow[dr, "y"] & G_{i+1} \arrow[r] \arrow[dr, "y"] & G_{i+2} \arrow[r] \arrow[dr, "y"] & \cdots \\
 & \oplus & \oplus & \oplus & \\
\cdots \arrow[r] & G_{i-1} \arrow[r, "(-1)^{i-1}\psi_{i-1}"'] & G_i \arrow[r] & G_{i+1} \arrow[r] & \cdots.
\end{tikzcd}$$

From this exact sequence, we get a long exact sequence of homologies

$$\cdots \to H^{i-1}(\mathcal{G}) \xrightarrow{y} H^{i-1}(\mathcal{G}) \to H^i(\mathcal{F} \otimes \mathcal{G}) \to H^i(\mathcal{G}) \xrightarrow{y} \cdots.$$

Here note that $H^{i-1}(\mathcal{G}) = H^i(\mathcal{G}[-1])$.

Proposition 2.22. *If $N = N' \oplus N''$, then $\bigwedge N = \bigwedge N' \otimes \bigwedge N''$. If $x' \in N$ and $x'' \in N''$, take $x = (x', x'') \in N$, then $K(x) = K(x') \otimes K(x'')$.*

Proof. Note that here the (skew-commutative) algebra structure of $\bigwedge N' \otimes \bigwedge N''$ is given by

$$(a \otimes b) \wedge (a' \otimes b') = (-1)^{\deg a' \deg b}((a \wedge a') \otimes (b \wedge b'))$$

for homogenous elements. This is just linear algebra. It suffices to check the differentials coincide, that is, for $y' \in \bigwedge N'$, $y'' \in \bigwedge N''$, $x \wedge (y' \otimes y'') = (x' \otimes 1 + 1 \otimes x'') \wedge (y' \otimes y'') = (x' \wedge y') \otimes y'' + (-1)^{\deg y'} y' \otimes (x'' \wedge y'')$. □

Corollary 2.23. *If $y_1, \dots, y_r$ are elements in $(x_1, \dots, x_n)$ and M is an R-module, then*

$$H^*(M \otimes K(x_1, \dots, x_n, y_1, \dots, y_r)) \simeq H^*(M \otimes K(x_1, \dots, x_n)) \otimes \bigwedge R^r$$

as graded modules, which means that

$$H^i(M \otimes K(x_1, \dots, x_n, y_1, \dots, y_r)) \simeq \bigoplus_{j+k=i} H^j(M \otimes K(x_1, \dots, x_n)) \otimes \bigwedge^k R^r.$$

So $H^i(M \otimes K(x_1, \dots, x_n, y_1, \dots, y_r)) = 0$ iff $H^j(M \otimes K(x_1, \dots, x_n)) = 0$ for any $i - r \le j \le i$.

Proof. As $y_1, \dots, y_r$ are elements in $(x_1, \dots, x_n)$, there is an isomorphism

$$R^n \oplus R^r \simeq R^n \oplus R^r$$

sending $(x_1, \dots, x_n, y_1, \dots, y_n)$ to $(x_1, \dots, x_n, 0, \dots, 0)$. So by functoriality of Koszul complex,

$$\begin{aligned} K(x_1, \dots, x_n, y_1, \dots, y_r) &\simeq K(x_1, \dots, x_n, 0, \dots, 0) \\ &\simeq K(x_1, \dots, x_n) \otimes K(0, \dots, 0). \end{aligned}$$

Here

$$K(0, \dots, 0) : 0 \to R \xrightarrow{0} \bigwedge^2 R^r \xrightarrow{0} \dots \xrightarrow{0} \bigwedge^r R^r \to 0.$$

□

Corollary 2.24. *If $x = (x', y) \in N = N' \oplus R$, then $K(x)$ is isomorphic to the mapping cone of $K(x') \xrightarrow{y} K(x')$. In particular, we have a long exact sequence*

$$\cdots \to H^i(M \otimes K(x')) \xrightarrow{y} H^i(M \otimes K(x')) \to H^{i+1}(M \otimes K(x)) \to$$
$$\to H^{i+1}(M \otimes K(x')) \xrightarrow{y} H^{i+1}(M \otimes K(x')) \to \dots$$

Proof. Note that $N' \oplus R \simeq R \oplus N'$. Hence $K(x) \simeq K(y, x') = K(y) \otimes K(x')$. This gives a short exact sequence

$$0 \to K(x')[-1] \to K(x) \to K(x') \to 0.$$

Tensoring with M, we get

$$0 \to M \otimes K(x')[-1] \to M \otimes K(x) \to M \otimes K(x') \to 0.$$

(Why exact?). □

2.5. **Proof of the main theorems.** The following is a more precise version.

Corollary 2.25. *If $x_1, \dots, x_i$ is an M-sequence, then*

$$H^i(M \otimes K(x_1, \dots, x_n)) = ((x_1, \dots, x_i)M : (x_1, \dots, x_n))/(x_1, \dots, x_i)M.$$

In particular, in this case, $H^j(M \otimes K(x_1, \dots, x_n)) = 0$ for $j < i$. If $IM \neq M$ ($I = (x_1, \dots, x_n)$) and $x_1, \dots, x_i$ is a maximal M-sequence, then $H^i(M \otimes K(x_1, \dots, x_n)) \neq 0$.

Proof. We do induction on i. If $i = 0$ this is trivial. If $i > 0$, then we do induction on n. If $n = i$, this follows easily by Example 2.14. If $n > i$, then by Corollary 2.24, there is an exact sequence

$$H^{i-1}(M \otimes K(x_1, \dots, x_{n-1})) \to H^i(M \otimes K(x_1, \dots, x_n)) \to$$
$$\to H^i(M \otimes K(x_1, \dots, x_{n-1})) \xrightarrow{x_n} H^i(M \otimes K(x_1, \dots, x_{n-1}))$$

Here by induction,

$$H^{i-1}(M \otimes K(x_1, \dots, x_{n-1})) = ((x_1, \dots, x_{i-1})M : (x_1, \dots, x_{n-1}))/(x_1, \dots, x_{i-1})M = 0$$

as x_i is not a zeo-divisor of $M/(x_1, \dots, x_{i-1})M$ (this also proves the second statement). Hence $H^i(M \otimes K(x_1, \dots, x_n))$ is just the kernel of

$$H^i(M \otimes K(x_1, \dots, x_{n-1})) \xrightarrow{x_n} H^i(M \otimes K(x_1, \dots, x_{n-1})).$$

By induction,

$$H^i(M \otimes K(x_1, \dots, x_{n-1})) = ((x_1, \dots, x_i)M : (x_1, \dots, x_{n-1}))/(x_1, \dots, x_i)M,$$

so it easy to compute the kernel.

To show the last statement, note that I is contained in the set of zero-divisors on $M/(x_1, \dots, x_i)M$, so I is contained in the union of associated primes and hence $I \subset \mathrm{ann}(x)$ for some non-zero $x \in M/(x_1, \dots, x_i)M$ by Corollary 1.12. This implies that $((x_1, \dots, x_i)M : I)/(x_1, \dots, x_i)M \neq 0$. □

Proof of Theorem 2.15. Let $y_1, \dots, y_s$ be a maximal M-sequence and r be the minimal such that

$$H^r(M \otimes K(x_1, \dots, x_n)) \neq 0.$$

The goal is to show that $r = s$.

By Corollary 2.23, r is the minimal such that

$$H^r(M \otimes K(x_1, \dots, x_n, y_1, \dots, y_s)) \neq 0.$$

If $IM \neq M$, then by Corollary 2.25, $r = s$. So it suffices to show that $IM \neq M$. This follows from Lemma 2.26(2) and the nonvanishing of homologies. □

Lemma 2.26. (1) *If $y \in (x_1, \dots, x_n)$, then $H^j(M \otimes K(x_1, \dots, x_n))$ is annihilated by y for any M and any j.*

(2) *If $(x_1, \dots, x_n)M = M$, then $H^j(M \otimes K(x_1, \dots, x_n)) = 0$ for any j.*

Proof. (1) Here we give a different proof from the book (which uses dual Koszul complex). Note that by Corollary 2.24, there is a long exact sequence

$$H^j(M \otimes K(x_1, \dots, x_n, y)) \to H^j(M \otimes K(x_1, \dots, x_n)) \xrightarrow{y} H^j(M \otimes K(x_1, \dots, x_n)).$$

So the statement is equivalent to that the first arrow is surjective. By the proof of Corollary 2.23, this arrow splits.

(2) Replacing R by $R/\mathrm{ann}(M)$ will not change $M \otimes K(x_1, \dots, x_n)$, so we may assume that $\mathrm{ann}(M) = 0$. By $(x_1, \dots, x_n)M = M$ and Lemma 1.2, there is $y \in (x_1, \dots, x_n)$ such that $(1 - y)M = 0$, which implies that $y = 1 \in (x_1, \dots, x_n)$. Then apply (1). □

Proof of Theorem 2.17. We prove the first statement by induction on n. Suppose $H^k(M \otimes K(x_1, \dots, x_n)) = 0$, then by Corollary 2.24,

$$H^{k-1}(M \otimes K(x_1, \dots, x_{n-1})) \xrightarrow{x_n} H^{k-1}(M \otimes K(x_1, \dots, x_{n-1}))$$

is surjective. Then by Nakayama's lemma, $H^{k-1}(M \otimes K(x_1, \dots, x_{n-1})) = 0$. By induction, $H^j(M \otimes K(x_1, \dots, x_{n-1})) = 0$ for $j \leq k-1$. By the long exact sequence in Corollary 2.24, $H^j(M \otimes K(x_1, \dots, x_n)) = 0$ for $j \leq k - 1$.

We prove the second statement by induction on n. Suppose $H^{n-1}(M \otimes K(x_1, \dots, x_n)) = 0$, then as above, $H^{n-2}(M \otimes K(x_1, \dots, x_{n-1})) = 0$, which implies that $x_1, \dots, x_{n-1}$ is an M-sequence by induction. Then by Corollary 2.25,

$$0 = H^{n-1}(M \otimes K(x_1, \dots, x_n)) = ((x_1, \dots, x_{n-1})M : (x_1, \dots, x_n))/(x_1, \dots, x_{n-1})M,$$

which implies that x_n is not a zero-divisor of $M/(x_1, \dots, x_{n-1})M$. □

3. Dimensions and depths

In this section we introduce fundamental theory on dimension and depth, which are basic invariants measuring size of a ring or an ideal.

3.1. **Dimension theory.** Recall that the *length* of a chain $P_r \supset P_{r-1} \supset \cdots \supset P_0$ is r.

Definition 3.1.
(1) The *(Krull) dimension* $\dim R$ of a ring R is defined to be the supremum of the lengths of chains of prime ideals in R.
(2) The *dimension* of an ideal I is $\dim I = \dim R/I$.
(3) The *codimension* of an ideal I is $\operatorname{codim} I = \min_{\text{pirme } P \supset I} \dim R_P$.

Remark 3.2. It is clear that $\dim I + \operatorname{codim} I \leq \dim R$. It is not always true that

$$\dim I + \operatorname{codim} I = \dim R.$$

For example, consider $R = k[x, y, z]/(xy, xz)$ and $I = (x - 1)$, then R corresponds to the union of a line ($x = 0$) and a plane ($y = z = 0$), and I corresponds to a point $(1, 0, 0)$. In this case, $\dim R = 2$, $\dim I = 0$, $\operatorname{codim} I = 1$. So we need to require some irreducibility for the equality to be true.

Theorem 3.3. *Let R be a domain finitely generated over a field, then*
(1)
$$\dim R = \text{tr.deg}_k R = \text{tr.deg}_k \text{Frac}(R).$$
(2) $\dim R$ *equals to the length of any maximal chains of prime ideals.*
(3)
$$\dim I + \operatorname{codim} I = \dim R.$$

Idea of proof. The proof uses the Noether normalization theorem: if $P_r \supset P_{r-1} \supset \cdots \supset P_0$ a maximal chain (in the sense that one cannot interesest in any more primes), then there exists a subring $k[x_1, \ldots, x_r] \simeq S \subset R$ such that R is integral over S and $P_i \cap S = (x_1, \ldots, x_i)$.

This implies that

$$\dim R = r = \text{tr.deg}_k S = \text{tr.deg}_k R.$$

For (2) $\Longrightarrow$ (3), we leave to exercise. □

Theorem 3.4 (Equivalent definitions for dimension of a local ring)**.** *Let $(R, \mathfrak{m}, k)$ be a local ring. Then* $\dim R$ *is equal to the following values:*
(1) *The minimal number d such that there exists elements $f_1, \ldots, f_d \in \mathfrak{m}$ not contained in any other primes in R (such $f_1, \ldots, f_d$ is called a system of parameters.);*
(2) $\dim R$ *equals to the length of any maximal chains of prime ideals.*
(3) $1+\deg(\dim_k(\mathfrak{m}^n/\mathfrak{m}^{n+1}))$*, here* $\dim_k(\mathfrak{m}^n/\mathfrak{m}^{n+1})$ *coincides with a polynomial in n if $n >> 0$.*

3.2. **Hilbert fuctions/polynomials.** Here we explain more about the Hilbert function/polynomial. Consider the polynomial ring $S = k[x_1, \ldots, x_n]$ and a finitely generated graded S-module $M = \bigoplus_{i \in \mathbb{Z}} M_i$ (Recall that "graded" means that $fM_i \subset M_{i+d}$ if f is homogenous of degree d). Then we can consider the Hilbert function $H_M(d) = \dim_k M_d$ (Why finite?).

Lemma 3.5. *There exists d_0 such that $H_M(d)$ is a polynomial in d if $d \geq d_0$.*

Proof. We do induction on n. If $n = 0$ this is trivial ($H_M(d) = 0$ if $d >> 0$). If $n > 0$, then consider the multiplication map

$$0 \to K_d \to M_d \xrightarrow{x_n} M_{d+1} \to C_d \to 0.$$

Then $K = \bigoplus_{i\in\mathbb{Z}} K_i$ and $C = \bigoplus_{i\in\mathbb{Z}} C_i$ are finitely generated graded S-modules. As the multiplications of x_n on K, C are 0, K, C are actually finitely generated graded $S/(x_n)$-modules. By dimension computing, we have

$$H_M(d+1) - H_M(d) = H_C(d) - H_K(d).$$

RHS is a polynomial for $d \geq d_0$ by induction hypothesis. So $H_M(d)$ is a polynomial for $d \geq d_0$. □

To conclude that $\dim_k(\mathfrak{m}^n/\mathfrak{m}^{n+1})$ coincides with a polynomial in n if $n >> 0$, we apply this lemma to $M = \bigoplus_{i\geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$.

3.3. **Regular local rings.** We first give some useful corollaries.

Corollary 3.6. *Let $(R, \mathfrak{m}, k)$ be a local ring. Then* $\dim R \leq \dim_k \mathfrak{m}/\mathfrak{m}^2$.

Proof. By Nakayama's lemma, $\dim_k \mathfrak{m}/\mathfrak{m}^2$ is the number of a minimal set of generators of $\mathfrak{m}$. □

Corollary 3.7. *Let R be ring and $I = (x_1, \dots, x_r) \neq R$. If P is minimal among all primes containing I, then* $\operatorname{codim} P \leq r$. *In particular,* $\operatorname{codim} I \leq r$.

Proof. Apply Theorem 3.4 to R_P. □

Corollary 3.8. *Let $(R, \mathfrak{m})$ be a local ring and $x \in \mathfrak{m}$ not a zero-divisor. Then* $\operatorname{codim}(x) = 1$ *and* $\dim R/(x) = \dim R - 1$.

Proof. By Corollary 3.7, $\operatorname{codim}(x) \leq 1$. If $\operatorname{codim}(x) = 0$, then (x) is contained in a minimal prime, which implies that x is a zero-divisor (Remark 1.11), a contradiction.

By definition, $d = \dim R/(x) \leq \dim R - \operatorname{codim}(x) = \dim R - 1$. On the other hand, if $\bar{x}_1, \dots, \bar{x}_d$ is a system of parameters of $\dim R/(x)$, then $(x, x_1, \dots, x_r) \subset \mathfrak{m}$ is not contained in other primes, so $\dim R \leq d + 1$. □

Definition 3.9. A local ring $(R, \mathfrak{m}, k)$ is *regular* if $\dim R = \dim_k \mathfrak{m}/\mathfrak{m}^2$, or equivalently, $\mathfrak{m}$ is generated by $d = \dim R$ elements $f_1, \dots, f_d$ (called a *regular system of parameters*). A ring is regular if its localization at every prime is regular.

Example 3.10. $k[x_1, \dots, x_n]$ is regular, $k[x, y]/(x^2 - y^3)$ is not regular.

The following tells that a regular system is actually a regular sequence.

Corollary 3.11. *Let $(R, \mathfrak{m}, k)$ be a regular local ring and $f_1, \dots, f_d$ a regular system of parameters, then $f_1, \dots, f_d$ is a regular sequence.*

Proof. We prove by induction on i that (1) $R/(f_1, \dots, f_i)$ is a regular local ring and $\dim R/(f_1, \dots, f_i) = d - i$, (2) f_{i+1} is not a zero-divisor on $R/(f_1, \dots, f_i)$.

Note that (1) holds for $i = 0$ By the next corollary, a regular local ring is a domain, so if (1) holds for i, then (2) holds for i.

Finally, if (2) holds for i, then (1) holds for $i+1$ by Corollary 3.8, as $\dim R/(f_1, \ldots, f_{i+1}) = \dim R/(f_1, \ldots, f_i) - 1 = d - i - 1$ and its maximal ideal is generated by $d - i - 1$ elements. □

Corollary 3.12. *Let $(R, \mathfrak{m}, k)$ be a regular local ring. Then R is a domain.*

Proof. We do induction on $d = \dim R$. If $d = 0$, then $\mathfrak{m} = 0$ and R is a field. If $d > 0$, then $\mathfrak{m} \neq \mathfrak{m}^2$ and $\mathfrak{m}$ is not minimal. So we can find $x \in \mathfrak{m}$ not in $\mathfrak{m}^2$ and not in any minimal primes of R (Why?). Consider $S = R/(x)$. Then $\dim S < \dim R$ and $\dim S \geq \dim R - 1$, so $\dim S = \dim R - 1$. Take $\mathfrak{n} = \mathfrak{m} \cap S$. Note that $\mathfrak{n}/\mathfrak{n}^2 = \mathfrak{m}/(\mathfrak{m}^2 + (x)) \subset \mathfrak{m}/\mathfrak{m}^2$ is a proper subspace, it can be generated by $d-1$ element, so S is regular of dimension $d-1$. By induction hypothesis, S is a domain. So (x) is prime. There exists a minimal prime $Q \subsetneq (x)$. For any $y \in Q$, $y = ax$ and $x \notin Q$, so $a \in Q$. This implies that $Q = xQ$, so $Q = 0$ by Nakayama's lemma. □

3.4. **Depth versus codimension, Cohen–Macaulay rings.**

Proposition 3.13. *Let R be a ring and I an ideal. Then* $\operatorname{depth}(I, R) \leq \operatorname{codim} I$.

The geometric meaning of this proposition is easy to understand: if $V(I)$ is contained in r hypersurfaces intersecting "properly", then its codimension is at most r.

Proof. Let $x_1, \ldots, x_r$ be a maximal regular sequence in I. Since x_1 is a non-zero-divisor, x_1 is not contained in any minimal primes, so $\operatorname{codim} I/(x_1) \leq \operatorname{codim} I - 1$. By induction, $\operatorname{codim} I/(x_1) \geq \operatorname{depth}(I/(x_1), R/(x_1)) = n - 1$. □

So it is interesting to investigate the equality case.

Definition 3.14. R is a *Cohen–Macaulay* ring if $\operatorname{depth}(I, R) = \operatorname{codim} I$ for every proper ideal I.

Theorem 3.15. *R is Cohen–Macaulay iff* $\operatorname{depth}(P, R) = \operatorname{codim} P$ *for every maximal ideal P.*

Proof. It suffices to show that if $\operatorname{depth}(P, R) = \operatorname{codim} P$ for every maximal ideal P, then $\operatorname{depth}(I, R) \geq \operatorname{codim} I$.

We first show that $\operatorname{depth}(I, R)$ can be localized, that is, there exists a maximal ideal P such that $\operatorname{depth}(I, R) = \operatorname{depth}(I_P, R_P)$. Using the Koszul complex (Theorem 2.15), $\operatorname{depth}(I, R)$ is the minimal integer r such that $H^r(K(x_1, \ldots, x_n)) \neq 0$, where $I = (x_1, \ldots, x_n)$, so there exists a maximal ideal P such that $H^r(K(x_1, \ldots, x_n))_P \neq 0$, which implies that $\operatorname{depth}(I, R) = \operatorname{depth}(I_P, R_P)$.

So after localization, we may assume that (R, P) is a local ring.

If P is the only prime containing I, then $\operatorname{codim} P = \operatorname{codim} I$ by definition. We claim that $\operatorname{depth} P = \operatorname{depth} I$. It suffices to show that $\operatorname{depth} P \leq \operatorname{depth} I$. As R/I is a local ring which has only one prime P, it can be shown that $P^k \subset I$ for some integer k (consider the radical of 0). Let $x_1, \ldots, x_r$ be a maximal regular sequence in P, then $x_1^k, \ldots, x_r^k \in I$, which is also a regular sequence (see Exercise). So $\operatorname{depth} P \leq \operatorname{depth} I$.

Suppose that P is the only prime containing I. By the Noetherian induction, we may assume that I is maximal among those satisfying $\operatorname{depth}(I, R) < \operatorname{codim} I$. We can take an element $x \in P$ but not in any minimal primes containing I, then $\operatorname{depth}(I + (x), R) = \operatorname{codim}(I + (x)) \geq \operatorname{codim} I + 1$. So we finish the proof by showing $r = \operatorname{depth}(I + (x), R) \leq \operatorname{depth}(I, R) + 1$. Suppose $I = (x_1, \dots, x_n)$ and $I + (x) = (x_1, \dots, x_n, x)$. By the Koszul complex (Theorem 2.15), $H^j(K(x_1, \dots, x_n, x)) = 0$ for $j < r$, which implies that $H^j(K(x_1, \dots, x_n)) = 0$ for $j < r - 1$ by Corollary 2.24 and Nakayama's lemma, so $\operatorname{depth}(I, R) \geq r - 1$. □

Finally we prove a property of CM ring.

Theorem 3.16. *Let $(R, \mathfrak{m})$ be a local ring and $x \in \mathfrak{m}$ is not a zero-divisor. Then R is CM iff $R/(x)$ is CM.*

Proof. Note that R is CM iff $\operatorname{depth} P = \dim R$, and $\dim R = \dim R/(x) + 1$. So it suffices to show that $\operatorname{depth}(P, R) = \operatorname{depth}(P/(x), R/(x)) + 1$. It is clear (Why?). □

4. Minimal resolutions and Auslander–Buchbaum formula

4.1. Free resolutions.

Definition 4.1 (Projective/free resolution)**.** A *projective resolution* of an R-module M is a complex of projective modules

$$\mathcal{F} : \cdots \to F_n \to \cdots \to F_1 \xrightarrow{\phi_1} F_0$$

which is exact and $\operatorname{coker}(\phi_1) = M$. Sometimes we also denote this by

$$\mathcal{F} : \cdots \to F_n \to \cdots \to F_1 \xrightarrow{\phi_1} F_0 (\to M \to 0).$$

$\mathcal{F}$ is a *free resolution* if all F_i are free. The length of $\mathcal{F}$ is the maximal n such that $F_n \neq 0$ (may be ∞).

Roughly speaking, F_0 gives information of generators of M, F_1 gives information of relations among generators, and so on.

Definition 4.2 (Projective dimension, global dimension)**.** The *projective dimension* $\operatorname{pd}(M)$ is defined to be the minimum of the lengths of projective resolutions of M. The *global dimension* $\operatorname{gldim}(R)$ is the supremum of $\operatorname{pd}(M)$ for all R-module M.

Theorem 4.3 (Auslander)**.** $\operatorname{gldim}(R) \leq n$ *iff* $\operatorname{pd}(M) \leq n$ *for any finitely generated M.*

So it suffices to consider only finitely generated M.

For local rings (and graded rings), the notion of projective modules and free modules coincides.

Theorem 4.4. *Let M be a finitely generated module over a Noetherian ring R. Then TFAE:*

(1) *M is projective.*
(2) *M_P is free for any maximal ideal P.*

Example 4.5 (Koszul complex and free resolution)**.** If $x_1, \ldots, x_n$ is a regular sequence of R, then $K(x_1, \ldots, x_n)$ is a free resolution of $R/(x_1, \ldots, x_n)$ (every higher homology vanishes). In particular, if $(R, \mathfrak{m})$ is a regular local ring, take $x_1, \ldots, x_n$ to be a minimal set of generators of $\mathfrak{m}$, then $K(x_1, \ldots, x_n)$ is a free resolution of $k = R/\mathfrak{m}$.

4.2. **Minimal free resolutions.** Now we turn to the notion of minimal free resolution. An economic way to construct free resolution is to take minimal generators each time.

Definition 4.6 (Minimal free resolution)**.** Let $(R, \mathfrak{m})$ be a local ring. A complex

$$\mathcal{F} : \cdots \to F_n \xrightarrow{\phi_n} F_{n-1} \to \ldots$$

is *minimal* if $\phi_n(F_n) \subset \mathfrak{m}F_{n-1}$.

The following lemma tells that this definition coincides with what we described above, that is, minimal resolution means taking minimal generators.

Lemma 4.7. *A free resolution*

$$\mathcal{F} : \cdots \to F_n \xrightarrow{\phi_n} F_{n-1} \xrightarrow{\phi_{n-1}} F_{n-2} \cdots \to F_1 \xrightarrow{\phi_1} F_0 (\xrightarrow{\phi_0} M \to 0).$$

over a local ring is minimal iff for each n, a basis of F_{n-1} maps onto a minimal set of generators of $\mathrm{coker}(\phi_n) = \mathrm{im}(\phi_{n-1})$.

Proof. Consider the maps

$$F_n \xrightarrow{\phi_n} F_{n-1} \to \mathrm{coker}(\phi_n)$$

and

$$F_{n-1}/\mathfrak{m}F_{n-1} \to \mathrm{coker}(\phi_n)/\mathfrak{m}\mathrm{coker}(\phi_n).$$

This is surjective and the kernel is $\mathrm{im}(\phi_n)/\mathfrak{m}F_{n-1}$. So $\mathcal{F}$ is minimal iff the above map is isomorphism, iff a basis of $F_{n-1}/\mathfrak{m}F_{n-1}$ maps onto a basis of $\mathrm{coker}(\phi_n)/\mathfrak{m}\mathrm{coker}(\phi_n)$. Then use Nakayama's lemma. □

Also one might wonder that minimal resolution has minimal length. This is true by the following corollary.

4.3. **Minimal free resolutions versus projective dimensions.**

Corollary 4.8. *Let $(R, \mathfrak{m}, k)$ be a local ring and M a finitely generated R-module. Then* $\mathrm{pd}(M)$ *is the length of every minimal free resolution of M. Furthermore,* $\mathrm{pd}(M)$ *is the smallest i such that* $\mathrm{Tor}^R_{i+1}(k, M) = 0$*. In particular,* $\mathrm{gldim} R = \mathrm{pd}(k)$.

Proof. Take i_0 the minimal such that $\mathrm{Tor}^R_{i+1}(k, M) = 0$.

Take a free resolution of M of length $\mathrm{pd}(M)$, then $\mathrm{Tor}^R_{i+1}(k, M) = 0$ for $i \geq \mathrm{pd}(M)$. So $\mathrm{pd}(M) \geq i_0$.

On the other hand, suppose that

$$\mathcal{F} : 0 \to F_n \xrightarrow{\phi_n} F_{n-1} \to F_{n-2} \cdots \to F_1 \xrightarrow{\phi_1} F_0 (\to M \to 0).$$

is a free resolution of length n, then $n \geq \mathrm{pd}(M) \geq i_0$.

$\mathcal{F}$ is minimal iff $k \otimes \mathcal{F}$ has 0 differentials, which implies that $\mathrm{Tor}^R_{i+1}(k, M) = k \otimes F_{i+1}$. In this case $\mathrm{Tor}^R_{i+1}(k, M) = 0$ iff $F_{i+1} = 0$ iff $i \geq n$. So $\mathcal{F}$ is minimal iff $n = i_0 = \mathrm{pd}(M)$.

For the last statement, it suffices to show that $\mathrm{pd}(k) \geq \mathrm{pd}(M)$ for every M, which is equivalent to $\mathrm{Tor}^R_{i+1}(k, M) = 0$ for $i \geq \mathrm{pd}(k)$. This can be proved by taking a free resolution of k. □

Corollary 4.9. *If R is a regular local ring of dimension n, then* $\mathrm{gldim}(R) = n$. *That is, every finitely generated module has a free minimal resolution of length $\leq n$.*

Proof. It suffices to show that $\mathrm{pd}(k) = n$. Take $x_1, \dots, x_n$ to be a minimal set of generators in $\mathfrak{m}$, it is a regular sequence by Corollary 3.11, then we saw that $K(x_1, \dots, x_n)$ is a minimal (Why? Check!) free resolution of length n. □

Remark 4.10 (Local rings vs graded rings). Let $R = \bigoplus_{i\geq 0} R_i$ a graded ring finitely generated over a field R_0 and $\mathfrak{m} = \bigoplus_{i\geq 1} R_i$ the homogenous maximal ideal. Then all results holds for $(R, \mathfrak{m})$ and graded R-module M.

Corollary 4.11 (Hilbert syzygy theorem). *Let k be a field. Every finitely generated graded module of $k[x_1, \dots, x_n]$ has a graded free minimal resolution of length $\leq n$.*

By the virtue of the Hilbert syzygy theorem, we can compute the Hilbert polynomial of a graded module M by $H_M(d) = \sum_i (-1)^i H_{F_i}(d)$, where $\mathcal{F} \to M$ is a graded free minimal resolution.

4.4. **Auslander–Buchbaum formula.** Finally, we introduce the Auslander–Buchbaum formula connecting projective dimension and depth.

Theorem 4.12 (Auslander–Buchbaum formula). *Let $(R, \mathfrak{m})$ be a local ring and M a finitely generated R-module of finite projective dimension. Then*

$$\mathrm{pd}(M) = \mathrm{depth}(\mathfrak{m}, R) - \mathrm{depth}(\mathfrak{m}, M).$$

If R is regular, then $m = (x_1, \dots, x_n)$ is generated by a regular sequence (Corollary 3.11). $\mathrm{depth}(\mathfrak{m}, R) = n$. $K(x_1, \dots, x_n)$ is a free resolution of k, so $\mathrm{Tor}^R_{i+1}(k, M) = H^{n-i-1}(M \otimes K(x_1, \dots, x_n))$. Corollary 4.8 says that $\mathrm{pd}(M)$ is the minimal i such that $\mathrm{Tor}^R_{i+1}(k, M) = 0$. Theorem 2.15 says that $\mathrm{depth}(\mathfrak{m}, M)$ is the minimal r such that $H^r(M \otimes K(x_1, \dots, x_n)) \neq 0$. So the equality follows.

Here note that a finitely generated M is not always of finite projective dimension, e.g. k for non-regular local ring (Theorem 4.13).

Proof. We do induction on $\mathrm{pd}(M)$. If $\mathrm{pd}(M) = 0$, then M is free and it's clear.

If $\mathrm{pd}(M) > 0$, consider $0 \to N \to F \to M \to 0$ be the first step of a minimal free resolution. Then $\mathrm{pd}(N) = \mathrm{pd}(M) - 1$. So by induction hypothesis, it suffices to show that $d := \mathrm{depth}(\mathfrak{m}, N) = \mathrm{depth}(\mathfrak{m}, M) + 1$.

Take $x_1, \dots, x_n$ to be generators of $\mathfrak{m}$, then consider the Koszul complex $K(x_1, \dots, x_n) = K(x)$. We have a long exact sequence

$$\cdots \to H^{i-1}(F \otimes K(x)) \to H^{i-1}(M \otimes K(x)) \to H^i(N \otimes K(x)) \to H^i(F \otimes K(x))$$

Since N and F have depth $\geq d$, $H^i(F \otimes K(x)) = H^i(N \otimes K(x)) = 0$ for $i < d$, which implies that $H^i(M \otimes K(x)) = 0$ for $i < d - 1$. To prove $\mathrm{depth}(\mathfrak{m}, M) = d - 1$, it remains to show that $H^{d-1}(M \otimes K(x)) \neq 0$. As

$H^d(N\otimes K(x))\neq 0$, it suffices to show that $H^d(N\otimes K(x))\to H^d(F\otimes K(x))$ is zero map.

If $\operatorname{pd}(N)>0$, then $\operatorname{depth}(\mathfrak{m},F)=\operatorname{depth}(\mathfrak{m},R)=d+\operatorname{pd}(N)>d$, so $H^d(F\otimes K(x))=0$. If $\operatorname{pd}(N)=0$, then N is free. So this map becomes $N\otimes H^d(K(x))\to F\otimes H^d(K(x))$ where $\operatorname{im}(N)\subset \mathfrak{m}F$ by minimality of free resolution. So this map is zero as $H^d(K(x))$ is annihilated by $\mathfrak{m}$ by Lemma 2.26. □

We close this lecture by the following application of the Auslander–Buchbaum formula.

Theorem 4.13. *A local ring has finite global dimension iff it is regular.*

Proof. We only need to show that if $(R,\mathfrak{m},k)$ has finite global dimension, then it is regular.

Let $x_1,\ldots,x_n$ be a minimal set of generators of $\mathfrak{m}$. From the principal ideal theorem, $\dim R\leq n$. It suffices to show that $\dim R\geq n$.

By Proposition 3.13, it suffices to show that $\operatorname{depth}(\mathfrak{m},R)\geq n$. By the Auslander–Buchbaum formula,

$$\operatorname{pd}(k)=\operatorname{depth}(\mathfrak{m},R)-\operatorname{depth}(\mathfrak{m},k)=\operatorname{depth}(\mathfrak{m},R).$$

It suffices to show that $\operatorname{pd}(k)\geq n$, that is, the minimal free resolution of k has length at most n. So we can consider the Koszul complex $K(x_1,\ldots,x_n)$, which has length n. The problem here is that it is not a resolution (not exact), unless R is regular (Example 4.5). So we need to compare $K(x_1,\ldots,x_n)$ with minimal free resolutions of k, and show that the length of any minimal free resolutions of k (which is $\operatorname{pd}(k)$) is at least n. This is done by the following lemma. □

Lemma 4.14. *Let $(R,\mathfrak{m},k)$ be a local ring and $x_1,\ldots,x_n$ a minimal set of generators of $\mathfrak{m}$. Then $K(x_1,\ldots,x_n)$ is a subcomplex of the minimal free resolution of k.*

Remark 4.15. In fact, all minimal free resolutions are isomorphic to each other.

Proof. Let

$$\mathcal{F}:\cdots\to F_1\to F_0$$

be the minimal free resolution of k. Then there is a comparison map of complexes $\phi: K(x_1,\ldots,x_n)\to\mathcal{F}$, namely,

$$\begin{tikzcd}
K(x): & 0 \arrow[r]\arrow[d] & \bigwedge^0 R^n \arrow[r]\arrow[d, "\phi_n"] & \ldots \arrow[r]\arrow[d] & \bigwedge^n R^n \arrow[r]\arrow[d, "\phi_0"] & k \arrow[r]\arrow[d, "="] & 0 \\
\mathcal{F}: & \ldots \arrow[r] & F_n \arrow[r] & \ldots \arrow[r] & F_0 \arrow[r] & k \arrow[r] & 0
\end{tikzcd}$$

Here the map $\phi_i:\bigwedge^{n-i}R^n\to F_i$ is defined by lifting

$$\begin{tikzcd}
\bigwedge^{n-i} R^n \arrow[r, "d"]\arrow[d, dotted, "\phi_i"] & \bigwedge^{n-i+1} R^n \arrow[d, "\phi_{i-1}"] \\
F_i \arrow[r, "\psi"] & F_{i-1}
\end{tikzcd}$$

as $\phi_{i-1}d(\bigwedge^{n-i}R^n) \subset \mathrm{im}\psi$ and $\bigwedge^{n-i}R^n$ is free. It suffices to show that ϕ_i splits. For $i = 0, 1$ this is trivial, in which case ϕ_i is isomorphic, as $x_1, \dots, x_n$ is a minimal set of generators.

Suppose that ϕ_{i-1} splits, it suffices to show that

$$R/\mathfrak{m} \otimes \phi_i : R/\mathfrak{m} \otimes \bigwedge^{n-i} R^n \to R/\mathfrak{m} \otimes F_i$$

is injective, because then by Nakayama's lemma the basis of $\bigwedge^{n-i}R^n$ maps to a subset of a basis of F_i, which gives the splitting naturally. Note that the image of d is in $\mathfrak{m}(\bigwedge^{n-i+1}R^n)$, d induces a map

$$\bar{d} : R/\mathfrak{m} \otimes \bigwedge^{n-i} R^n \to \mathfrak{m}/\mathfrak{m}^2 \otimes \bigwedge^{n-i+1} R^n.$$

As $R/\mathfrak{m} \otimes \phi_{i-1}$ is injective by induction hypothesis, it suffices to show $\bar{d}$ is injective. This is linear algebra and we omit the proof. □

References

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